Kinematics, Rotations and Euler Angles

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Introduction
- Common Assumptions
- Convenient Vector Formulation

Direction Cosine Matrix

Euler Angles

Euler Rotation Matrices
- 2D & 3D
- General 3D Rotation Matrix

Kinematics

Basic Dynamics

Common Assumptions

Newton's law's are only valid when written relative to an non-accelerating/non-rotating reference frame. in practice unless your planning on going into space the flat earth model is often used.

1. The earth is flat, stationary and therefore an approximate inertial reference frame. Acceleration of gravity is constant and perpendicular to the surface of the earth
2. The atmosphere is at rest relative to the ground (zero wind)
3. The aircraft is conventional with fixed thrust source and a right left plane of symmetry
4. Modeled as a variable mass particle
5. Forces are symmetric in flight and act at the center of gravity.

Convenient Vector Formulation

Used in AUV's and UAV's the following state space representation is very convent

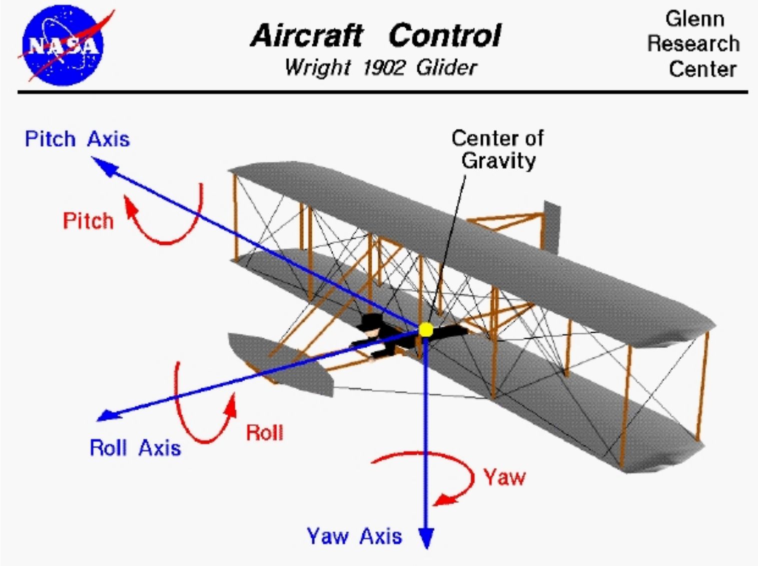
The UAV's position is expresses relative to an inertial reference.

$$\eta_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \ \eta_2 = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \ \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{bmatrix}$$

Also the ν vector holds the linear and angular velocity

$$\nu_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \ \nu_2 = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \ \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}$$

Aircraft Axis System (North, East, Down)

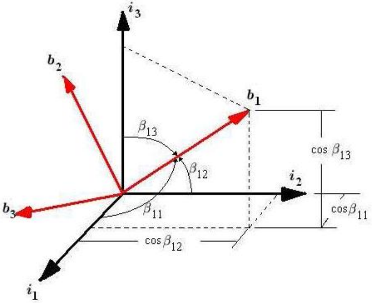


The Direction Cosine Matrix (DCM) a 3D rotation Matrix

A general matrix to convert from one orientation to another in the same location

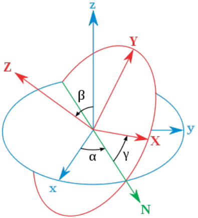
If we express unit vectors of b as a projection of the inertial reference frame we get

$$[b] = \begin{bmatrix} \cos\beta_{11} & \cos\beta_{12} & \cos\beta_{13} \\ \cos\beta_{21} & \cos\beta_{22} & \cos\beta_{23} \\ \cos\beta_{31} & \cos\beta_{32} & \cos\beta_{33} \end{bmatrix} [i]$$



The Euler Angle System

- Euler angles are the standard way of thinking of orientation in 3D and is rather intuitive.
- Euler angles are an ordered set of rotation applied in the order of Yaw, Pitch and Roll for aircraft.
- Changing the order will result in a different attitude being represented. Initially unseen there is a singularity in the representation.



Applying Rotations (Video)

The Euler Angle limitations

Due to the order of rotation as pitch approaches 90° it becomes aligned with the yaw axis and a degree of freedom is lost.This is known as gimbal lock since a degree of freedom is lost since two axises of rotation line up.
Rotating through 90° will cause roll and yaw to flip 180°.

Gimbel lock in Computer Graphics explained (video)

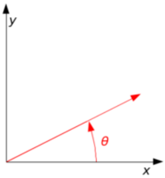
2D Rotation Matrix

The rotation matrix is defend by the relation below

$V_{local} = M * V_{Earth}$

Properties

- Square matrix
- determinant $= 1$
- $M^T = M^{-1}$



What is a 2x2 rotation Matrix for this rotation

General form

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(cc rotation by θ)

Basic Roll, Pitch & Yaw Rotations in 3D

There are three basic rotation matrices in three dimensions:

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \tag{1}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2}$$

$$R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

General Rotation Matrix

GO!
The order of the three rotations Roll, Pitch, Yaw applied from the left − z, y, x
result of the z,y

$$R_z(\psi)R_y(\theta) = \begin{bmatrix} \cos\psi\cos\theta & -\sin\psi & \cos\psi \\ \sin\psi\cos\theta & \cos\psi & \sin\psi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{4}$$

General Rotation Matrix $R_z(\psi)R_y(\theta)R_x(\phi) =$

$$\begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\theta + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

Transforming Angular Velocities

Using the three major rotations again and applying them to angular velocities

$$\dot{\eta_2} = [R_2(\eta_2)]\nu_2$$

$$\dot{\eta_2} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + [R_x(\phi)]^T \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + [R_x(\phi)]^T [R_y(\theta)]^T \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = [R_2(\eta_2)]^{-1} \dot{\eta_2}$$

taking the inverse you are left with

$$[R_2(\eta_2)] = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\psi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\psi/\cos\theta \end{bmatrix}$$

Formalizing the Kinematics

Applying the first R matrix it is possible to determine the change in location based on the current linear velocity and orientation.

$\dot{\eta}_1 = R_1(\eta_2)\nu_1$
where $R_1 =$

$$\begin{bmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\theta + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

Implementing the second rotation matrix the change in orientation can be determined from the angular velocities

$\dot{\eta}_2 = R_2(\eta_2)\nu_2$
where

$$R_2(\eta_2) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\psi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\psi/\cos\theta \end{bmatrix}$$

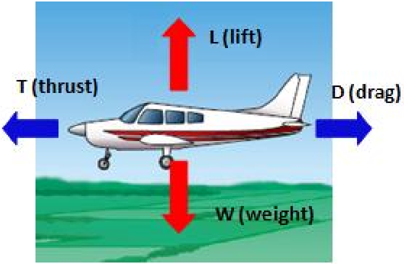
Basic Fixed Wing Dynamics

$\mathbf{F} = m\mathbf{a}$ Forces are

made up of vectors Thrust, Aerodynamic Forces & weight

$\mathbf{F} = \mathbf{T} + \mathbf{A} + \mathbf{W}$

Aerodynamic forces are the vector sum of Lift and Drag vectors.
$\mathbf{A} = \mathbf{L} + \mathbf{D}$



In level non-accelerating flight
$\mathcal{T} = \mathcal{D}$ & $\mathcal{L} = \mathcal{W}$

Center of Gravity

The aircraft is assumed to be a rigid body with all forces acting through the center of gravity or CG