A. Nassirharand
Assistant Professor
Department of Mechanical Engineering
University of Kentucky,
Lexington Ky 40506-0046

J. H. Taylor
General Electric Corporate Research
and Development,
Control Technology Branch,
Schenectady, N Y 12345

K. N. Reid
Professor and Dean,
College of Engineering, Architecture
and Technology
Oklahoma State University,
Stillwater Okla 74078

Controller Design For Nonlinear Systems Based on Simultaneous Stabilization Theory and Describing Function Models

A new systematic and algebraic linear control system design procedure for use with highly nonlinear plants is developed. This procedure is based on simultaneous stabilization theory and sinusoidal-input describing function models of the nonlinear plant, and is presently applicable to single-input single-output, time-invariant, deterministic, stable, and continuous-time systems which are representable in standard state-variable differential equation form. Three software utilities to implement the controller design procedure are also outlined. This method and the associated software is applied to a position control problem of the sort encountered in robotics, and the results are compared with those previously obtained using both linear and nonlinear PID control.

1 Introduction

Methods for the design of linear controllers for linear plants are well established It is well known, however, that a conventionally designed linear controller (e.g., a PID or lead/lag controller designed on the basis of a small-signal or Taylor series linear model via a standard approach such as root locus or Ziegler-Nichols tuning) may not produce adequate performance over a variety of operating regimes, especially if the plant is highly nonlinear [1-9] In such cases, it is not clear whether one needs a nonlinear controller, or if a different linear controller design method might suffice.

In this research, we considered the latter possibility, and developed a synthesis technique we call multi-range controller design. The term "multi-range" arises from the fact that the controller design is based on several plant characterizations that correspond to different operating regimes. These regimes are characterized not by operating point, but rather by the expected amplitude or range of the input signals. Once these models are obtained via the describing function approach, the design method proceeds in a systematic fashion and yields controller designs which are demonstrably better than those produced using traditional linear system methods.

Control engineers may utilize linear control theory (classical or modern), optimal control theory, or adaptive control theory to obtain a controller that will cause a plant to have satisfactory response. Systems with linear plants have received considerable attention, and effective results are available. In contrast, systematic controller design techniques for use with nonlinear plants are still in the early stages of development (for example, [1-9]). In the latter context, the use of existing linear control system design techniques based on small-signal linearization often leads to a design that must be "tuned" or otherwise modified in a nonsystematic fashion in order to make it work reliably. The use of controller design based on either optimal control theory or adaptive control is usually justified if the classical control theory is not applicable. However, such control laws are difficult or impossible to design for nonlinear plants, and, when they can be obtained, are usually very difficult to implement

Our primary objective is to present a new, systematic, algebraic linear controller design procedure for use with highly nonlinear plants. An algebraic procedure minimizes the degree of subjective judgment that has to be employed by the designer to arrive at a practical design; such a procedure can easily be automated on a digital computer. The secondary objectives are: (1) to develop appropriate numerical algorithms which can be used to implement the controller design procedure in a computer-aided engineering environment, and (2) to demonstrate the design procedure by applying it to a meaningful problem.

The controller synthesis procedure is composed of the following six steps:

1 Define the desired closed-loop system performance specifications, and identify the reference linear model whose static and dynamic behavior matches the desired closed-loop system performance specifications.

2 Characterize the input/output behavior of the nonlinear plant via describing function models for several amplitudes of input excitation (e g., small, medium, and large input signals).

3 Choose two describing function models and identify via a suitable fitting approach a corresponding set of linear systems.

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4 Determine the set of all linear controllers that simultaneously stabilize these linear systems.

5 Search this set for the minimum-sensitivity linear controller

6 Validate the design via digital simulation.

In some cases, the user may have to execute Step 1 after the input/output characterization task of Step 2, since it may not be possible to arrive at realistic performance requirements without a basic knowledge of the input/output behavior.

This procedure identifies a minimum-sensitivity linear controller through the use of simultaneous stabilization theory [10] and is based on sinusoidal-input describing function models of the plant. A *minimum-sensitivity* linear controller is defined herein as a controller which achieves a closed-loop system (i) whose dynamic behavior is relatively insensitive to the amplitude of the command signal, and (ii) whose dynamic behavior satisfies a set of user-defined performance measures as closely as possible. A robust design is achieved since item (i) above is an important consideration in robustness [1]. If the procedure is unable to identify a linear controller which achieves (i) and (ii) above, then a nonlinear controller design procedure may have to be used

The remainder of our development assumes that two operating regimes are sufficient to characterize the amplitude-dependence of the nonlinear plant for the purpose of controller design. These two regimes are usually selected based on describing function models obtained for more than two input amplitudes. If two regimes are not adequate, then one can in principle consider as many regimes and models as required; however, to the best of our knowledge, the characterization of all compensators that simultaneously stabilize more than two plants has not been computationally verified.

2 Controller Synthesis Procedure

Controller synthesis requires the following a priori information:

(i) The mathematical model of the nonlinear plant in the following state-variable differential equation form,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u(t)) \quad (1)$$

$$y(t) = g(\mathbf{x}(t), u(t)) \quad (2)$$

(ii) A number of operating regimes of the nonlinear plant. These operating regimes are defined by: (1) the range of the expected amplitude level of the excitation command, and (2) the range of the excitation frequency of interest.

Technical details of the six-step controller synthesis procedure are given below.

2.1 Performance Specifications. The first step of the design procedure is to define a set of performance specifications for the closed-loop system in either the time or frequency domain. In either domain, the user is then required to identify a linear system model whose dynamic behavior matches the specified performance specifications for the closed-loop system; this model we denote $G_r(s)$. Three possible strategies for identification of the reference linear model has been suggested [11] The simplest approach is to select a linear second-order system model which has the desired natural frequency and damping ratio; this strategy is used in the work presented herein. The user is not limited to identifying second-order models: Meyfarth [12] has presented step, impulse, and frequency response characteristics of linear third-order models in the form of dimensionless response plots for a number of different combinations of system parameters. These response plots may be examined visually to select the reference linear model.

2.2 Input/Output Characterization. It is desirable to use the powerful tools of linear systems theory in designing controllers for nonlinear systems. Characterizing the input/output behavior of the nonlinear plant in the frequency domain followed by applying a fitting procedure that approximates the frequency response by that of a linear system is an approach that permits the use of linear design methods.

There are two basic techniques for obtaining such characterizations. The first and most common approach is to linearize the equations of motion around an operating point of interest. This practice is often referred to as small-signal linearization (SSL). The input/output relation is obtained by replacing each nonlinearity with a linear term whose gain is the slope of the nonlinearity at the operating point The SSL method has the following disadvantages: it is not applicable to nonlinear plants which have discontinuous or multivalued nonlinearities (e.g., saturation, hysteresis, and backlash), it eliminates the dependence of the input/output behavior of a nonlinear plant on the amplitude of the input signal, and SSL models of nonlinear plants are usually highly sensitive to the assumed operating point. Perturbation of the nonlinear system from its assumed operating point may result in unsatisfactory system behavior.

The second approach for characterizing the input/output behavior of a nonlinear plant does not have these disadvantages. Instead, one replaces each nonlinearity with a

Nomenclature

- a = the amplitude of the excitation signal
- $a_0, a_1, \ldots, a_p$ = the coefficients of the numerator polynomial of transfer function $G_r(s)$ (see equation (7))
- $b_0, b_1, \ldots, b_q$ = the coefficients of the denominator polynomial of transfer function $G_r(s)$ (see equation (7))
- $C(s)$ = a stabilizing compensator (may be subscripted for clarity)
- $d(s)$ = a transfer function ($d(s)$ and $n(s)$ are coprime factors of $G(s)$; $G(s) = n(s)/d(s)$) (may be subscripted for clarity)
- DFGEN = describing function GENerator (a software utility)
- DRLCD = dual-range linear controller design (a software utility)
- e = error signal
- $e(\omega)$ = frequency-domain error function
- $\mathbf{f}$ = function representing the time rate of change of state variables (nonlinear dynamics of system to be controlled)
- f_c = Coulomb friction coefficient ($f_c = 10$ Nm)
- f_v = viscous friction coefficient ($f_v = 0.1$ Nm-s/rad)
- F = objective function for minimum-sensitivity linear controller design
- g = function representing the system output
- $G(j\omega)$ = a frequency response function (may be subscripted for clarity)
- $G_{df}(j\omega; a, u_0)$ = frequency response data representing a SIDF model
- J = moment of inertia
- m_1, m_2 = torque-motor gain constants
- $n(s)$ = a transfer function (see $d(s)$ above)
- $N(a)$ = an amplitude-dependent gain (describing function)

quasilinear gain which does depend on the amplitude of the excitation signal. The function corresponding to the quasilinear gain is referred to as the describing function of that nonlinearity; it is based on the form of the excitation signal which is assumed in advance. There is an abundant literature on this approach (cf., [2, 13, 14]), so we will not develop it in detail. In summary, sinusoidal-input describing function models (SIDF models) have been shown to be meaningful for controller design applications [5], especially with regard to the issue of robustness. SIDF models of nonlinear plants do not have the disadvantages of the small-signal models, and they enjoy the following properties: They may be used to interface with tools of linear control system analysis and design, they approach the corresponding SSL models (if they exist) when the amplitude of the excitation signal is small (so the control engineer should not obtain results that are inconsistent with standard SSL methods), and SIDF models can be obtained for any plant that can be represented in state-variable form. Most significantly, SIDF models provide an excellent basis for a robust control system design because they retain the amplitude sensitivity characteristics of the nonlinear plant.

SIDF models of the nonlinear plant at a given operating point are obtained by determining the gain and phase of the nonlinear system response to a sinusoidal input at set of discrete frequencies. There are two approaches for obtaining SIDF models: The first approach is similar to that used in limit cycle analysis [13-16]; it involves replacing each nonlinearity with a gain of known form but unknown value and solving the set of nonlinear algebraic equations that corresponds to harmonic balance The second approach utilizes direct simulation and evaluation of Fourier integrals. The first method assumes the input to each nonlinearity is nearly sinusoidal; this assumption is not required in the second approach. Algorithms and software for the second approach have recently been developed [4, 11]; the routine DFMOD [4] was created as part of the GE computer-aided control engineering environment, while DFGEN (See Appendix D of [11]) was developed to automate the generation of SIDF models as a part of the first author's research effort. This method proceeds as follows: The plant is excited by a sinusoid,

$$u(t) = u_0 + a\cos(\omega t) \tag{3}$$

where u_0 is the DC value of the input signal and a is the amplitude of the excitation signal. Then, the dynamic equations of motion, equations (1, 2), are numerically integrated to obtain the output $y(t)$ as a function of time. Fourier integrals for period k are calculated simultaneously, once $y(t)$ has reached steady-state. These integrals are given by

$$I_{m,k} = \int_{(k-1)T}^{kT} y(t)e^{-jm\omega t}dt \tag{4}$$

where $k = 1, 2, \ldots, m = 0, 1, 2, \ldots$, and $T = 2\pi/\omega$. The constant or DC component of the response is given by $I_{0,k}$, and the SIDF transfer function at discrete frequencies, which is represented by the complex number $G_{df}(j\omega; u_0, a)$, is given by

$$G_{df}(j\omega; u_0, a) = G_{1,k} = \omega I_{1,k}/a\pi \tag{5}$$

In order to analyze the importance of higher harmonic effects, one may also evaluate

$$G_{m,k}(jm\omega; u_0, a) = \omega I_{m,k}/a\pi, \quad m = 2, 3, \ldots \tag{6}$$

The above technique for generation of SIDF models is restricted to stable plants; otherwise, evaluation of the Fourier integrals would be meaningless. For a given excitation amplitude a, equation (5) is evaluated at discrete frequencies (over the frequency range of interest to the user) to obtain one quasilinear model of the nonlinear plant. This procedure may be repeated for various excitation amplitudes to obtain a number of quasilinear models of the nonlinear plant. In our procedure, the two quasilinear models whose gain characteristics enclose those of all others in the class are selected, and they are set aside for system identification of the next step.

2.3 Linear System Identification. We must next identify two linear system models whose dynamic behavior approximates that of the two selected SIDF models from the previous step. This must be done before the class of all stabilizing controllers can be parameterized via the application of simultaneous stabilization theory The user has to exercise judgment in this step: it may be desirable to obtain a different quality of fit at different frequency; e.g., to sacrifice the quality of fit at high frequency to obtain a better fit near the gain and phase cross-over points Another consideration is stability: since these models must approximate the input/output model of the nonlinear plant in the specified operating regimes, they must be constrained to be stable. Other issues relating to this process which will be discussed in Section 3.3 where an example problem is treated.

A new method for identifying single-input single-ouput linear systems that fit frequency response data was developed

Nomenclature (cont.)

- $p(s)$ = a transfer function ($p(s)$ and $q(s)$ are coprime, and, together with $n(s)$ and $d(s)$, satisfy the Bezout identity; see equation (10) (may be subscripted for clarity)
- $q(s)$ = a transfer function (see $p(s)$ above)
- $r(s)$ = a transfer function (the parameter in parameterizing the class of all stabilizing controllers)
- $\tilde{r}(s)$ = a transfer function (the parameter in parameterizing the class of all controllers that stabilize $r(s)$)
- s = Laplace transform variable
- SYSID = SYStem IDentifier (a software utility)
- t = time
- T = period of a periodic signal
- T_e, T_m = torques (see Fig. 1)
- $u(t)$ = system input
- u_0 = DC component of the input signal
- V_{in} = torque-motor input excitation voltage
- $W(\omega)$ = a frequency-domain weighing function
- WMSE = weighted mean-square error
- $\mathbf{x}(t)$ = a vector representing system states
- $\dot{\mathbf{x}}(t)$ = a vector representing the time rate of change of vector $\mathbf{x}(t)$
- $\dot{x}_1(t)$ = the controlled variable, position (see Fig. 1)
- $\dot{x}_2(t)$ = velocity (see Fig. 1)
- $y(t)$ = system output
- α = a weighing coefficient
- δ = the break-point (gain-change point) of a limiter
- ζ = damping ratio
- ω = frequency (rad/s)
- ω_n = natural frequency
- $\Sigma(C(s), G(s))$ = a closed-loop unity-feedback system whose forward-path transfer function is $C(s)G(s)$; $\Sigma = CG/(1+CG)$

to meet these needs. The rational linear systems used for fitting may be represented via a transfer function of the following form:

$$G_f(s) = \frac{\sum_{i=0}^{p} a_i s^i}{\sum_{k=0}^{q} b_k s^k} \tag{7}$$

The objective is to identify p, q, a_i, and b_k in such a manner that the frequency response of the identified transfer function $G_f(s)$ approximates the given SIDF frequency response data with suitable fidelity To do this, we developed an algorithm that produces minimum weighted mean-square error (WMSE), defined by

$$\text{WMSE} = \int_{\omega} W(\omega) e(\omega) d\omega \tag{8}$$

where $W(\omega)$ is the weighting function and

$$e(\omega) = |G_f(j\omega) - G_{df}(j\omega;\, u_0,\, a)|^2. \tag{9}$$

For a given p and q, we use Chandler's [17] implementation of the Hooke and Jeeves [18] pattern direct search algorithm to minimize WMSE. A starting solution is obtained from the application of the system identification technique of Lin and Wu [19] which uses a generalized least-squares technique [20] to identify linear systems of the general form given by equation (7) with $b_0 = 1$. This algorithm was reformulated to allow identification of linear systems of the general form given by equation (7) with either $a_0 = 1$ or $b_0 = 1$ [4, 11]; this was necessary to allow fitting type-one systems A command-driven software utility SYSID was developed based on the above system identification technique (see Appendixes A and E of [11])

The primary advantages of the above system identification technique with respect to the system identification technique of Lin and Wu are: (1) the user is able to obtain a better quality of fit in specific frequency ranges, and (2) the user is able to perform constrained minimization; in particular, it is necessary for our approach to restrict the optimization routine to obtain stable solutions (approximations that have no right-half s-plane poles)

2.4 Controller Parameterization. Next, we parameterize all controllers that simultaneously stabilize the closed-loop systems corresponding to the two linear system models of the nonlinear plant from the previous step This involves developing a mathematical expression for all stabilizing controllers in terms of the linear system models from the previous step and another as yet unknown stable nth-order transfer function denoted $r(s)$. The term *parameterization* arises from the fact that the class of all stabilizing controllers are expressed in terms of the *parameter* $r(s)$. Such a parameterization provides a logical means for obtaining the desired minimum-sensitivity controller. The following single-input/single-output version of the simultaneous stabilization theory of Vidyasagar and Viswanadham [10] was adapted to fulfill the purpose of this step of the design procedure.

Theorem 1: The class of all controllers that stabilize a plant $G(s)$ may be parameterized in terms of a stable linear transfer function, $r(s)$, as follows:

$$C(s) = \frac{p(s) + r(s)d(s)}{q(s) - r(s)n(s)} \tag{10}$$

is the stabilizing controller, where $\{n(s),\, d(s)\}$ are stable coprime factors of $G(s)$, and $\{p(s),\, q(s)\}$ are stable, coprime, and satisfy the Bezout identity: $p(s)n(s) + q(s)d(s) = 1$.

The unknown parameter $r(s)$ may be selected to achieve certain performance criteria. Note that the original setting of this theorem was more general: Instead of specifying that $r(s)$, $\{p(s),\, q(s)\}$, $\{p(s),\, q(s)\}$ and the resulting control system be stable, it is merely required that these transfer functions have poles in an arbitrary closed region of the s-plane [10]. This generality has not been used in this work, although it may be useful to do so in the future, since the specified region in the s-plane may be selected in terms of performance characteristics such as response speed and damping.

The parameterization of all controllers that stabilize two stable linear system models is given next. Let the pairs $\{n_i(s),\, d_i(s)\}$ correspond to the stable coprime factors of $G_i(s)$, $i=0$, 1, and designate the corresponding factors that satisfy the Bezout identify by the stable coprime pair $\{p_i(s),\, q_i(s)\}$.

Theorem 2: The set of all controllers that simultaneously stabilize $G_0(s)$ and $G_1(s)$ is given by

$$C_{ss}(s) = \frac{p_0(s) + \bar{r}(s)d_0(s)}{q_0(s) - \bar{r}(s)n_0(s)} \tag{11}$$

where $\bar{r}(s)$ belongs to the set of all *stable* controllers that stabilize $G_{ss}(s)$,

$$G_{ss}(s) = \frac{-n_0(s)d_1(s) + d_0(s)n_1(s)}{q_0(s)d_1(s) + p_0(s)n_1(s)} \tag{12}$$

The set of all controllers that stabilize $G_{ss}(s)$ is given by Theorem 1; the condition that $\bar{r}(s)$ must itself be stable (it is said that the plant G_{ss} must be *strongly* stabilizable) must be imposed in addition. See [10] for proofs

2.5 Controller Synthesis. This step involves searching the set of all stabilizing controllers which were parameterized in the previous step to obtain the *minimum-sensitivity linear controller*, i.e., the controller that produces a closed-loop system whose dynamic behavior most closely approximates that of the identified linear system $G_r(s)$ of Step 1.

Denote the two linear models from Step 3 by $G_0(s)$ and $G_1(s)$ Since these linear plants are stable, the stable coprime pairs $\{n_i(s), d_i(s)\}$ and $\{p_i(s), q_i(s)\}$ may be defined by

$$n_i(s) = G_i(s),\, d_i(s) = 1,\, p_i(s) = 0, \text{ and } q_i(s) = 1 \tag{14}$$

where $n_i(s)$ and $d_i(s)$ are the coprime factors of $G_i(s)$; $i=0$, 1, and $q_i(s)$ and $p_i(s)$ clearly satisfy the Bezout identity. Substitute (14) into (11); the class of all stabilizing controllers is

$$C_{ss}(s) = \frac{\bar{r}(s)}{1 - \bar{r}(s)G_0(s)} \tag{15}$$

where $\bar{r}(s)$ belongs to the set of all stable controllers that stabilize $G_{ss}(s)$ (equation (12)). From equations (12–14) we obtain

$$G_{ss}(s) = G_1(s) - G_0(s). \tag{16}$$

Since both $G_0(s)$ and $G_1(s)$ are proper and stable, their difference is also proper and stable. In this case, the stable coprime pairs $\{n_{ss}(s),\, d_{ss}(s)\}$ and $\{p_{ss}(s),\, q_{ss}(s)\}$ may be defined by

$$n_{ss}(s) = G_{ss}(s),\, d_{ss}(s) = 1,\, p_{ss}(s) = 0, \text{ and } q_{ss}(s) = 1 \tag{17}$$

where $\{n_{ss}(s),\, d_{ss}(s)\}$ are the coprime factors of $G_{ss}(s)$, and $\{q_{ss}(s),\, p_{ss}(s)\}$ satisfy the Bezout identity. Then, the set of all $\bar{r}(s)$ which corresponds to the class of all controllers that stabilize $G_{ss}(s)$ is given by Theorem 1:

$$\bar{r}(s) = \frac{r(s)}{1 - r(s)G_{ss}(s)}. \tag{18}$$

Finally, substitute (18) into (15) and use (16) to obtain the class of all controllers that stabilize both $G_0(s)$ and $G_1(s)$:

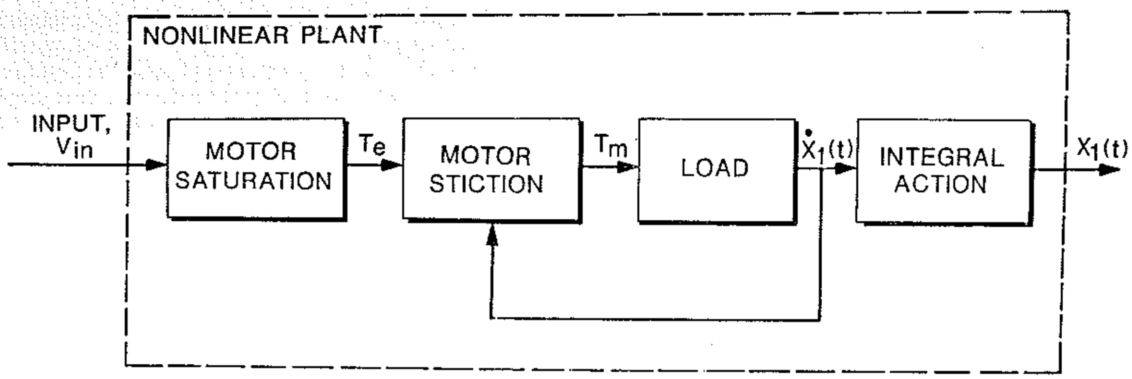

Fig. 1 Position servo open-loop model schematic

$$C_{ss}(s) = \frac{r(s)}{1-r(s)G_1(s)} \tag{19}$$

where

$r(s)$ is stable and, (19a)

$G_{ss}(s)$ must be strongly stabilizable (19b)

Therefore, all controllers that stabilize both $G_0(s)$ and $G_1(s)$ is given by (19) subject to constraints (19a) and (19b). This set must be searched for the minimum-sensitivity linear controller using the following algorithm:

Algorithm 1:

1 Assume a stable transfer function $C_{ss}(s)$,
2 Compute $r(s)$ using (19),
3 If constraints (19a) and (19b) are violated, provide a barrier for the optimization routine and go to Step 6; otherwise go to Step 4,
4 Compute objective function F:

$$F = \alpha F_0 + (1-\alpha)F_1;\ 0 \le \alpha \le 1.0 \tag{20}$$

Where

$$F_0 = |G_r(s) - \Sigma(C_{ss}(s), G_0(s))|^2, \tag{21}$$

$$F_1 = |G_r(s) - \Sigma(C_{ss}(s), G_1(s))|^2, \text{ and} \tag{22}$$

the notation $\Sigma(C(s), G(s)) = \dfrac{C(s)G(s)}{1+C(s)G(s)}$ is used.

5 If F is minimized then stop; otherwise go to Step 6,
6 Iterate the coefficients of $C_{ss}(s)$; then go to Step 2.

The user may use algorithm 1.1, given below, to arrive at a fair starting solution for $C_{ss}(s)$:

Algorithm 1.1:

The function $C_s(s)$,

$$C_s(s) = \frac{G_r(s)}{G_0(s)(1-G_r(s))} \tag{24}$$

where $G_r(s)$ and $G_0(s)$ are prespecified, may be used as a starting solution.

Note that the function $C_s(s)$ satisfies

$$G_r(s) = \frac{C_s(s)G_0(s)}{1+C_s(s)G_0(s)} \tag{25}$$

which establishes the relation of the starting solution to the user-defined reference model and one of the SIDF-fitted models. The command-driven Dual-Range Linear Controller Design (DRLCD) software utility (see Appendix F of [11]) was developed to automate the execution of Algorithms 1 and 1.1.

Algorithm 1 performs constrained minimization, which presents the following difficulties:

1 Unconstrained optimization algorithms that minimize a given function either based on function evaluations (direct search methods [21]) and/or based on evaluation of the gradients of the function [21, 22] are not effective here due to the presence of the constraints. In fact, no technique with a unique search direction may be applied: Once constraints are violated, the objective function is set to a large value, and the optimization algorithm stops after a few iterations.

2 The constraints are indirect, in the sense that there does not exist a function that characterizes the constraints. The constraints can only be imposed via tests; once they are failed the objective function is set to a large value. Therefore, few available constrained optimization algorithms can be applied

The simplex search method [23] was thus employed to overcome these problems, as recommended by Wright [24]. This method forms a regular simplex in the space of the independent variables. The objective function is evaluated at each vertex, and the vertex with the highest functional value is located and reflected through the centroid to complete a new simplex. As long as the performance index decreases smoothly, the iterations move along crabwise. This feature of the simplex algorithm avoids the forbidden zones of independent variables

3 Demonstration Problem

The following example [1] typifies a simple position control problem The block diagram of the open-loop system is shown in Fig. 1, and the mathematical model of the open-loop system (nonlinear plant) is given by equations (26)–(29):

$$\dot{x}_1 = x_2, \tag{26}$$

$$\dot{x}_2 = T_m/J \tag{27}$$

where $J = 0.01\ \text{kg}-\text{m}^2$ Servomotor saturation effects are modeled by

$$T_e = \begin{cases} m_1 V_{in} \text{ if } |V_{in}| < \delta \\ \text{Sign}(V_{in}) \cdot (m_1\delta + m_2(|V_{in}| - \delta)) \text{ if } |V_{in}| > \delta \end{cases} \tag{28}$$

where $\delta = 0.5$ volts, $m_1 = 5$ Nm/V, and $m_2 = 1.0$ Nm/V The servomotor friction characteristic includes Coulomb and viscous effects;

$$T_m = \begin{cases} T_e - f_v\dot{x}_1 - f_c \text{ Sign}(\dot{x}_1) \text{ if } |T_e| > f_c \\ T_e - f_v\dot{x}_1 - f_c \text{ Sign}(\dot{x}_1) \text{ if } x_1 \neq 0 \\ 0.0 \text{ if } |T_e| < f_c \text{ and } \dot{x}_1 = 0 \end{cases} \tag{29}$$

where $f_v = 0.1$ Nms/rad and $f_c = 1.0$ Nm.

The objective is to synthesize a linear controller for this nonlinear plant. The resulting feedback system is to satisfy a set of user-defined performance specifications with as little sensitivity to the input amplitude as possible. We proceed as in Section 2:

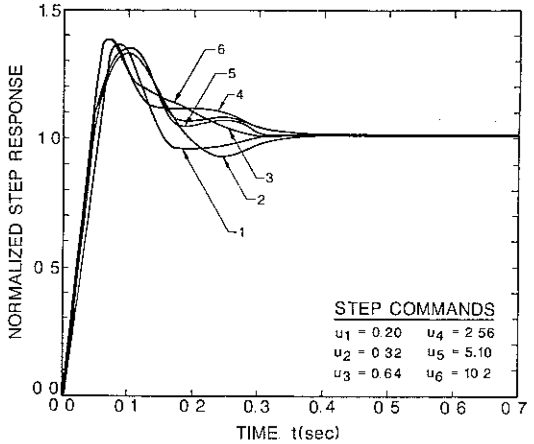

Fig. 2 Normalized step responses of the nonlinear PID and plant

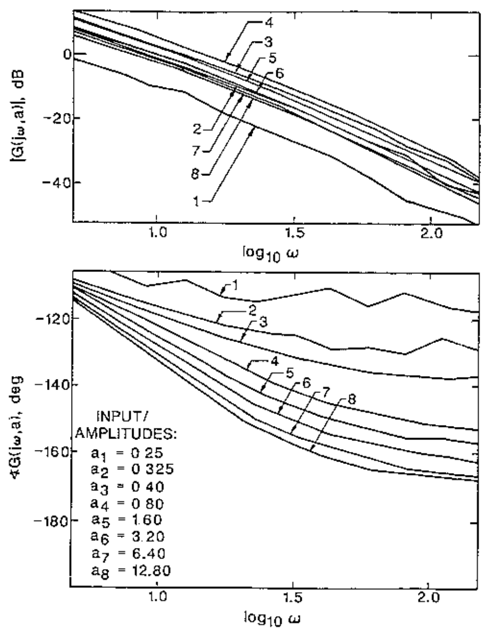

Fig. 3 SIDF models of the nonlinear plant

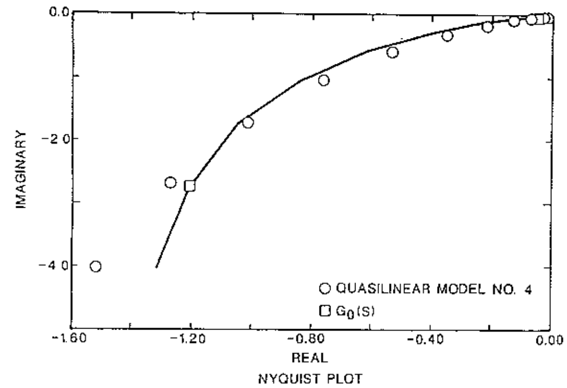

Fig. 4 Approximation of the frequency response of the quasilinear model number 4 with a linear model of an open-loop system

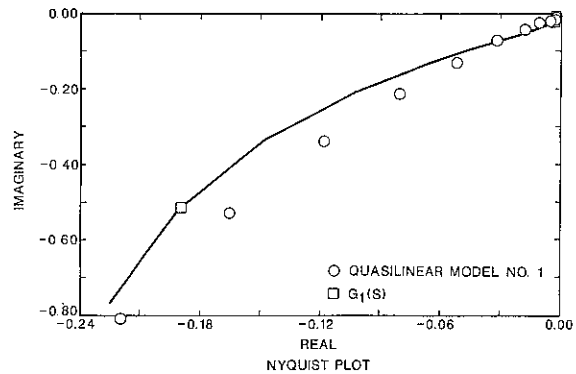

Fig. 5 Approximation of the frequency response of the quasilinear model number 1 with a linear model of an open-loop system

3.1 Performance Specifications. Taylor and Strobel [1] synthesized a nonlinear controller for the nonlinear plant of Fig. 1. The normalized step responses of the closed-loop nonlinear feedback system for a range of step amplitudes is shown in Fig. 2. The maximum percent overshoot is about 37, and the two percent settling time is about 0.30 seconds; this performance is roughly consistent with Ziegler-Nichols tuning. For comparison purposes, these time-domain performance specifications are utilized herein also. The third strategy of Section 2.1 is thus applicable in this case: A linear second-order model whose damping ratio and natural frequency are 0.375 and 37.0 radians/seconds exhibits approximately a 37 percent overshoot and a 2 percent settling time of 0.3 seconds when excited with a step input; therefore,

$$G_r(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \tag{30}$$

where $\omega_n = 37.0$ rad/s and $\zeta = 0.375$.

3.2 Input/Output Characterization. The input/output behavior of the nonlinear plant is to be characterized for several operating regimes via SIDF models obtained using the simulation technique discussed in Section 2.2. Taylor and Strobel [1] obtained SIDF models for eight input amplitude values; the resulting frequency response plots are depicted in Fig 3. In this example, the upper and lower bounds on the frequencies of interest were 5.0 and 150.0 radians/seconds; the input amplitudes considered were 0.25, 0.325, 0.40, 0.80, 1.60, 3.20, 6.40, and 12.8 volts. The two quasilinear models whose gain variation enclose those of the others are selected for system identification purposes, that is, the curves which are labeled by the numerals 1 and 4.

3.3 Linear System Identification. Linear systems must be identified approximating the input/output SIDF models designated in the previous step. These linear models are denoted by $G_0(s)$ and $G_1(s)$, respectively; they were identified via the computer-aided system identification software utility SYSID outlined in Section 2.3. The inputs to SYSID are the real and imaginary parts of the frequency response information for the quasilinear models, and the resulting linear approximations are

$$G_0(s) = \frac{1.231 + 0.003989\, s}{s(0.05475 + 0.003761\, s)}, \tag{31}$$

$$G_1(s) = \frac{1.000 + 0.04031\, s}{s(0.22521 + 0.23641\, s)}. \tag{32}$$

Comparisons of the two quasilinear models (from Step 2) with these linear models are shown in Figs. 4 and 5.

This example demonstrates several important considerations in the context of fitting SIDF models with linear system tranfer functions:

1 Overfitting: The most obvious concern in the fitting process is determining the most appropriate order of the transfer function being identified. In fitting data points with polynomials, it is well known that as the order of the polynomial is increased, the magnitude of the error at the

Table 1 Frequency response comparison for the SIDF model number 4 and the approximating open-loop system, $G_0(s)$

REAL PART			IMAGINARY PART		
QUASILINEAR MODEL	$G_0(s)$	%ERROR	QUASILINEAR MODEL	$G_0(s)$	%ERROR
-1.51700	-1.31700	13.23	-4.02500	-4 0450	0.501
-1.27000	-1.20800	4.95	-2.68300	-2.7360	1.993
-1.01200	-1.04700	3.46	-1.71800	-1.7560	2.190
-0.76310	-0.83910	9.96	-1.03900	-1.0500	1.091
-0.53320	-0.61330	15.02	-0.59120	-0.5800	1.891
-0.34820	-0.40910	17.50	-0.32070	-0.2990	6.772
-0.21590	-0.25280	17.12	-0.16940	-0.1483	12.430
-0.12820	-0.14790	15.39	-0.08635	-0.0738	14.520
-0.07161	-0.08359	16.73	-0.04423	-0.0384	13.210
-0.04051	-0.04625	14.17	-0.02152	-0.0215	0.324
-0.02272	-0.02529	11.28	-0.01114	-0.0130	6.490
-0.01141	-0 01373	20.31	-0 00504	-0.0084	66 720

Table 2 Frequency response comparison of the SIDF model number 1 and the approximating open-loop system, $G_1(s)$

REAL PART			IMAGINARY PART		
QUASILINEAR MODEL	$G_1(s)$	%ERROR	QUASILINEAR MODEL	$G_1(s)$	%ERROR
-0.21960	-0.19920	9.30	-0.80670	-0.79310	1.69
-0.16530	-0.18070	9.28	-0.52950	-0.52910	0.87
-0.11840	-0.14470	22.23	-0.33930	-0.34130	0.59
-0.08062	-0.10348	28.30	-0.21290	-0.21590	1.40
-0.05169	-0.06721	30.01	-0.13150	-0.13720	4.400
-0.03191	-0.04065	27.39	-0.07176	-0.89550	24.79
-0.01820	-0.02343	28.78	-0.04177	-0.06047	44.75
-0.01115	-0.01312	17.62	-0.02442	-0.04208	72.31
-0.00578	-0.00722	24.81	-0.01818	-0.02992	64.62
-0.00319	-0.00394	23.39	-0.00812	-0.02157	165.70
-0.00211	-0.00213	1.197	-0.00486	-0.01567	222.70
-0.00543	-0.00115	78 75	-0.00229	-0 01144	400 90

discrete data points will decrease but the behavior between points will not be smooth. In other words, the user must attend to the tradeoff between the overall quality of the fit and the error at discrete data points.

2 Nonlinear behavior: SIDF models do not represent rational linear system models, so the standard relation between the real and imaginary parts of the frequency response does not hold. Thus a perfect fit is not possible, and the quality of fit is highly dependent on the dominance of nonlinear effects. For example, motor stiction effects are more dominant for small signals than for larger signals, so a better quality fit is expected for high-amplitude excitation signals than at lower amplitude, as is clearly apparent from examination of the %ERROR columns of Tables 1 and 2.

3 Instability: We have found that the first two issues above often result in the identification of high-order transfer functions that are unstable, while the quasilinear models are representations of stable nonlinear systems. This must be avoided.

4 High-frequency effects: If the user supplies weighting factors that specify a low percent error at high frequency, then an overall low-quality fit will be obtained. It should be kept in mind that the quasilinear models are only an approximation to a nonlinear model which itself is an approximate mathematical description of a physcial process; over-emphasis on fitting accuracy where substantial roll-off has taken place is usually not warranted.

Therefore, linear model identification is probably the most demanding part of this controller synthesis procedure. The results of Step 6 (design validation) will reveal whether or not the final synthesized controller is acceptable; if the results are not acceptable, then either one must iterate on Step 3, or this method has failed and an *n-range* linear or nonlinear controller must be be synthesized.

3.4 Controller Parameterization. All controllers that stabilize the identified linear systems of the previous step must be parameterized. This parameterization was developed in

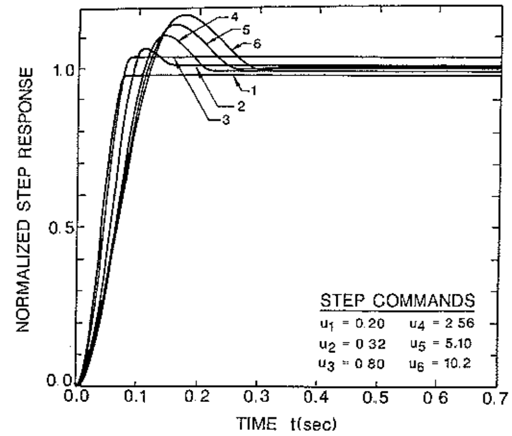

Fig. 6 Normalized step response plots of the synthesized linear controller and the nonlinear plant

Section 2.4, and it is given by (19) subject to constraint (19*a*) and (19*b*).

3.5 Controller Synthesis. The class of all stabilizing controllers must be searched for the minimum-sensitivity controller, i.e., the controller C_0 that causes the frequency-domain behavior of $\Sigma(C_0, G_0)$ and $\Sigma(C_0, G_1)$ to be as close to $G_r(s)$ as possible. Algorithm 1.1 is used to get a starting solution (corresponding to $\alpha = 1$), then Algorithm 1 is applied with various values of α ranging from 0 to 1 to obtain candidate linear controllers for the nonlinear plant. The performance of each candidate controller is evaluated by simulation (linear controller and nonlinear plant) to see if the dynamic behavior of the resulting closed-loop system is satisfactory. If adequate performance cannot be achieved, then the user should consider nonlinear compensator synthesis (e.g., [1, 5]).

Algorithm 1.1 yielded the following controller:

$$C_s(s) = \frac{74.954 + 5.149\ s}{34.166 + 1.342\ s + 0.004\ s^2} \tag{33}$$

This "starting controller" is in fact "simultaneously stabilizing"; also, it has a high-frequency pole that may be removed to obtain

$$C_{ss}(s) = \frac{59.970 + 4.119\ s}{27.750 + 1.000\ s} \tag{34}$$

Normalized step responses (actual step response divided by the reference input command) of the resulting closed-loop system are shown in Fig. 6. This figure shows that the closed-loop system satisfies the specified performance measures; i.e., percent overshoot is less than 37 and the 2 percent settling time is 0.3 seconds. In this case, no optimization was required.

In order to exercise the optimization algorithm, a different starting solution was selected by a major change in the denominator of C_{ss}:

$$C_{ss}(s) = \frac{59.970 + 4.119\ s}{1.000 + 1.000\ s} \tag{35}$$

From the application of Algorithm 1 with $\alpha = 0.5$, the synthesized controller is

$$C_{ss}(s) = \frac{0.789 + 17.626\ s}{8.495 + 4.133\ s} \tag{36}$$

Figure 7 depicts the performance of the nonlinear control system with the starting controller and with the optimized controller. The transient behavior with the optimized controller shows substantial improvement over that with the starting controller; however, the optimized system yields an infinite 2 percent settling time. Since this is caused by stiction,

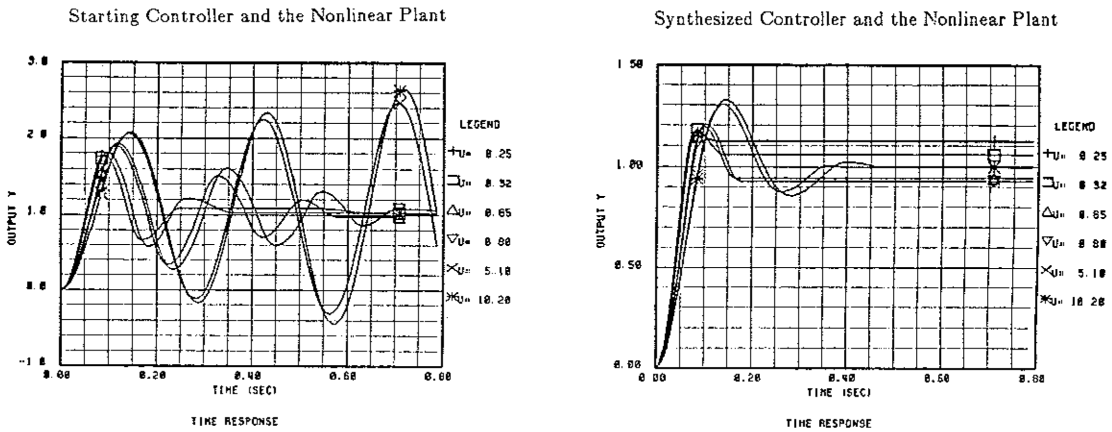

Fig. 7 Normalized step response of the linear controller and the nonlinear plant ($\alpha = 0.5$)

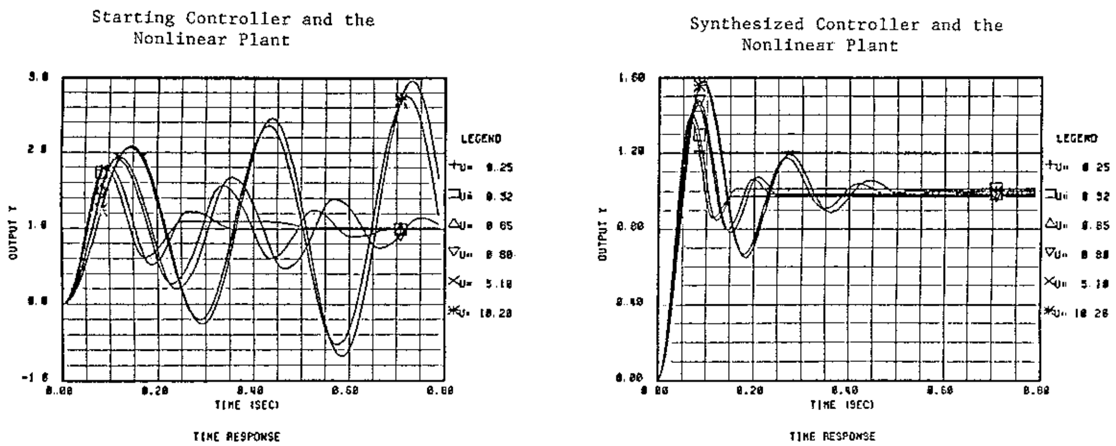

Fig. 8 Normalized step response plots of the linear controller and the nonlinear plant ($\alpha = 0.1$)

which is more pronounced for low amplitude excitation signals, Algorithm 1 is applied again with $\alpha = 0.1$. The resulting synthesized controller is

$$C_{ss}(s) = \frac{11.805 + 7.563\, s}{1.527 + 0.655\, s} \tag{37}$$

Figure 8 may be used to compare the performance of the starting controller in controlling the nonlinear plant output with that of the optimized controller. Again, in this case, the transient behavior with the optimized controller shows substantial improvement over that with the starting controller; however, the optimized system with high amplitude excitation signals yields a 59 percent overshoot which is 22 percent higher than that desired. This is due to the fact that $\alpha = 0.1$, and the response due to the low amplitude excitation signals is weighted nine times heavier than the response due to high amplitude excitation signals; for step magnitudes with very large magnitude integral windup causes excessive overshoot. Hence, α should be regarded as a tuning parameter which is to be set by the designer. If a satisfactory value of α is not obtained, then the user may have to design a nonlinear controller.

3.6 Controller Performance Comparisons

3.6.1 Dual-Range Linear Versus Single-Range Linear Controllers. A linear controller may be synthesized based on only one SIDF model of the nonlinear plant; this approach can be called Single-Range Linear Controller Design, SRLCD. SIDF model number 4 (See Fig. 3), which has the highest gain,

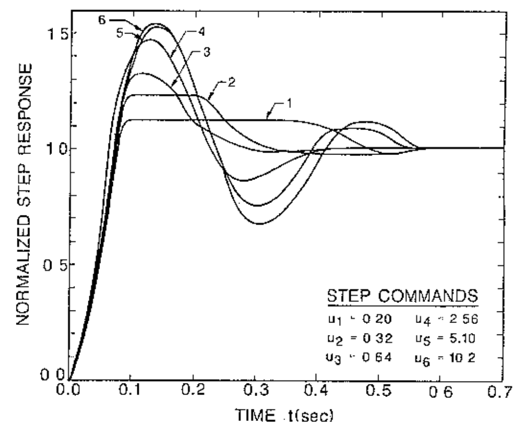

Fig. 9 Normalized step response plots of the system with linear PID control of the normalized plant

was selected by Taylor and Strobel [1] to design a linear PID for the example problem. The normalized step responses for a range of input amplitudes is shown in Fig. 9. From a comparison of Figs. 9 and 6, the following conclusions may be drawn:

1 The SRLCD linear PID controller produced a closed-loop system which is appreciably more sensitive to the amplitude level of the command signal than that produced by DRLCD, even though both techniques were attempting to meet the same

performance requirements. From this, we infer that the DRLCD controller design technique is preferable. In any case, a SRLCD design should precede the application of the DRLCD technique, so the user can determine if its use is justified.

2 The closed-loop system with the single-range linear PID controller is more sluggish than the closed-loop system with the dual-range linear controller. Also, the apparent degree of the closed-loop system stability with the single-range linear PID controller decreases as the amplitude-level of the command signal increases. (This effect is caused by integral wind-up, which could be corrected.) This is not the case with the closed-loop system with the dual-range linear controller.

3 The 2 percent settling time of the closed-loop system with the single-range linear PID ranges from 0.3 to 0.6 seconds over the range of amplitude of the excitation signal. The 2 percent settling time of the closed-loop system with the dual-range linear controller ranges from 0.08 to 0.3 seconds over the same range of amplitude-level of the excitation signal; again, this is a substantial improvement

3.6.2 Dual-Range Linear Versus Multi-Range Nonlinear Controller Design. A nonlinear PID controller was designed by Taylor and Strobel [1] based on eight quasilinear models of the nonlinear plant. The normalized step responses for a range of input amplitudes is shown in Fig. 2. From the the comparisons of Figs. 6 and 2 we conclude that the closed-loop system with the nonlinear controller and the closed-loop system with the optimized dual-range linear controller are approximately equally insensitive to the amplitude of the command signal with respect to the basic performance measures; however, the transient response waveform characteristics are more uniform for the nonlinear controller case

4 Summary and Conclusions

The goal of this study [11] was to develop a systematic controller synthesis procedure for the design of feedback control systems with highly nonlinear, single-input single-output, time invariant, continuous-time plants. The premise of this research was that linear approximations to SIDF models of a nonlinear plant can be used in conjunction with simultaneous stabilization theory to design effective control systems for highly nonlinear plants [2]. The work presented herein supports the conclusion that such a controller design procedure does indeed produce closed-loop systems that are "robust" in the sense of reduced amplitude sensitivity. Therefore, this goal has been met.

Two types of information are required for this controller synthesis procedure: (1) the mathematical model of the nonlinear plant is standard state-variable differential equation form, and (2) a set of performance measures in either the time or frequency domain. The mathematical model of the plant is used to generate the SIDF models of the plant at the specified operating regimes of interest. If such differential equation models are not readily available, the user can obtain the SIDF models via laboratory experiments, exciting the plant with sinusoidal inputs of various amplitudes over a number of frequencies and measuring the first-harmonic component of the response. The performance measures are used to generate a desired open-loop transfer function $G_r(s)$ that forms the basis for controller optimization.

A minimum-sensitivity linear controller is identified using the controller synthesis procedure given in Section 2. This procedure is both systematic and algebraic, so it may be employed without a high level of subjectivity. The only area calling for judgment is Step 3, where the user may have to iterate to obtain a suitable linear model fit.

The method and the associated software were applied to a position control problem of the sort encountered in robotics. The performance of the closed-loop system with the linear controller synthesized using the DRLCD procedure was found to be superior to that of the system with a linear PID controller based on one SIDF model, and it was comparable, in terms of the given performance specifications, to the performance of the system with a nonlinear controller.

There are several areas for future work that may prove to be fruitful:

1 Extension of the controller system design procedure from a two-model approach to an n-model approach; i.e., base the controller synthesis on $n(n>2)$ linear approximations to SIDF models of the nonlinear plant. This would handle cases in which the amplitude sensitivity of the nonlinear plant requires the use of more than two SIDF models.

2 Extension of the method and software utilities for use with multiple-input multiple-output nonlinear systems.

3 Extension of the method to permit the solution of the disturbance rejection problem.

4 Extension of Steps 4 and 5 of the design procedure to permit coprime factorization and optimization over a user-defined region of the s-plane rather than the open left half plane.

5 Refinement of the optimization algorithm, either by adapting or developing an optimization routine based on random search, or by adapting a multivariable constrained optimization algorithm (e.g. SUMT [25]). In the latter case, the user may define a constraint in relation to the "degree" by which the constraint (19*b*) is violated. This "degree" may be quantified by the length of the vector erected from the poles of $r(s)$ and $\tilde{r}(s)$ to the boundary of the user-defined region of the s-plane.

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