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Sequential design of decentralized control systems

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Our main result is a new sequential method for the design of decentralized control systems. Controller synthesis is conducted on a loop-by-loop basis, and at each step the designer obtains an explicit characterization of the class **C** of *all* compensators for the loop being closed that results in closed-loop system poles being in a specified closed region **D** of the *s*-plane, instead of merely stabilizing the closed-loop system. Since one of the primary goals of control system design is to satisfy basic performance requirements that are often directly related to closed-loop pole location (bandwidth, percentage overshoot, rise time, settling time), this approach immediately allows the designer to focus on other concerns such as robustness and sensitivity. By considering only compensators from class **C** and seeking the optimum member of that set with respect to sensitivity or robustness, the designer has a clearly-defined limited optimization problem to solve without concern for loss of performance. A solution to the decentralized tracking problem is also provided. This design approach has the attractive features of expandability, the use of only 'local models' for controller synthesis, and fault tolerance with respect to certain types of failure.

1. Introduction

Large-scale systems such as factory automation systems, power systems, large space structures, complex chemical processes, and data communication and control networks are generally spatially distributed. Centralized control of such systems is often uneconomical owing to the cost of information transfer between controllers, and unreliable because of the complexity of the information network. Therefore, such large systems are more likely to be controlled by several local controllers, with each controller measuring certain outputs of the system and generating only certain local inputs to the system. Thus, decentralization often arises as an important consideration in the design of strategies for controlling such systems, and the study of the stabilization and regulation of large systems using decentralized feedback is of immense practical interest.

The area of decentralized control of large systems has attracted much attention over the past few years. For instance, Wang and Davison (1973) introduced the notion of fixed modes under decentralized feedback conditions (or, more simply, decentralized fixed modes), and established that decentralized stabilization is possible only if the decentralized fixed modes are stable. Corfmat and Morse (1976) specified the necessary and sufficient conditions for spectrum assignment under decentralized feedback. Further, Vidyasagar and Viswanadham (1982) presented a frequency-domain characterization of decentralized fixed modes. The decentralized servo problem has received some treatment (Davison 1976, Viswanadham and Ramakrishna 1981). Davison and Ferguson (1981) have proposed parametric optimiza-

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tion methods for decentralized controller synthesis. Also, Bennet (1979) generalized Rosenbrock's INA method for block diagonally dominant systems, and Davison and Ozguner (1982) have proposed a sequential stabilization scheme based on state–space models. However, despite these contributions, it is fair to say that systematic procedures for designing decentralized controllers are presently unavailable.

We present a new sequential method of designing decentralized compensators. This approach is a generalization of the loop-by-loop design methods suggested by Rosenbrock *et al.* (1974). Since this loop-by-loop compensator synthesis is conducted using the coprime factorization approach (Desoer *et al.* 1980) our procedure is more algebraic than graphical. Further, for each loop, we characterize all stabilizing compensators, and in this way we deal with this critical design issue at the start rather than at the end of the design process.

The stabilization and regulation problems for centralized multivariable systems have been the staple of system theory for several years, and substantial progress has been made recently. Using the matrix fraction description of transfer function matrices, a characterization of all compensators that stabilize a given plant is now available (Desoer *et al.* 1980). In addition, a characterization of all tracking compensators has also been provided (Vidyasgar and Viswanadham 1982). We use these results in our design procedure.

Organization

In § 2, we review the existing results on decentralized control of large-scale systems and controller synthesis using coprime factorizations. In § 3, we develop the sequential method and prove the generalized decomposition theorem. We then present the sequential design algorithm for decentralized control in § 4. The design procedure yields a closed-loop system with poles in any desired closed region $\mathbf{D}$ of the s-plane, which in most practical applications satisfies a major objective of the design process. Additional specifications on sensitivity, disturbance rejection, robustness, etc. can be met by appropriate choice within the class of compensators. Section 5 treats the decentralized tracking problem. Finally, we conclude this paper with a comparison of our method with other results existing in the literature.

2. Background

In this paper, we consider the l-channel large-scale system described by

$$\begin{bmatrix} y_1(s) \\ y_2(s) \\ \vdots \\ y_l(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \dots & G_{1l}(s) \\ G_{21}(s) & G_{22}(s) & \dots & G_{2l}(s) \\ \vdots & \vdots & & \vdots \\ G_{l1}(s) & G_{l2}(s) & \dots & G_{ll}(s) \end{bmatrix} \begin{bmatrix} e_1(s) \\ e_2(s) \\ \vdots \\ e_l(s) \end{bmatrix} \tag{2.1}$$

or

$$y(s) \triangleq G(s)e(s) \tag{2.2}$$

where $G(s)$ in (2.1) is the transfer function matrix, and $y_i(s)$, $e_i(s)$; $i = 1, 2, \dots, l$ are the local output and input to the ith subsystem, of dimension q_i and m_i, respectively. Our aim is to design decentralized feedback controllers $C_1, C_2, \dots, C_l$ (see Fig. 1), i.e. to close the system loops via

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_l \end{bmatrix} = \begin{bmatrix} C_1 & 0 & 0 & \dots & 0 \\ 0 & C_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & C_l \end{bmatrix} \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \vdots \\ \bar{e}_l \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_l \end{bmatrix} \tag{2.3}$$

or

$$e \triangleq C(s)\bar{e} + u \tag{2.4}$$

where

$$\bar{e} = \bar{u} - y \tag{2.5}$$

and $\bar{u}$ and u are the reference input and disturbance vectors, respectively.

From (2.2), (2.4) and (2.5), we have the final input/output relation as

$$y(s) = [I + GC]^{-1} GC\bar{u} + [I + GC]^{-1} Gu \tag{2.6}$$

The *basic design problem* is to find *all* $C_1, C_2, \dots, C_l$ such that the closed-loop system in (2.6) has its poles in a desired region **D** of the s-plane. A typical region **D** is shown in Fig. 2, based on the classical performance constraints on speed and damping.

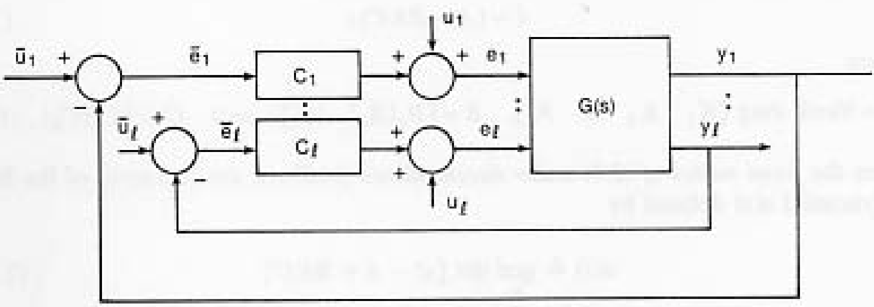

Figure 1. Decentralized control system configuration.

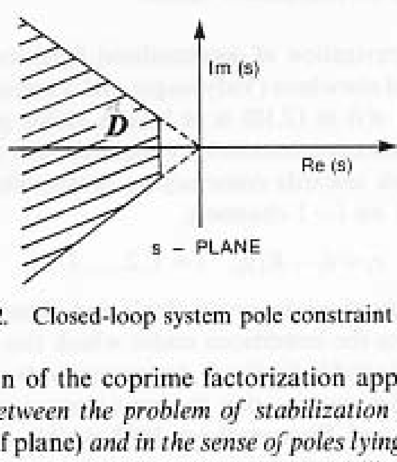

Figure 2. Closed-loop system pole constraint region **D**.

A major contribution of the coprime factorization approach is that *no real distinction need be made between the problem of stabilization in the standard sense* (i.e. poles lying in the left-half plane) *and in the sense of poles lying in a desired region* **D**. We shall thus often use the simpler terminology 's.able', 'stabilize', 'stabilization', etc., with the understanding that it may always be taken in the extended sense with the obvious changes in assumptions and conditions. We shall only comment on this generalization where required for clarity.

2.1. *Fixed modes and decentralized control*

The issue of finding necessary and sufficient conditions for decentralized stability was considered by Wang and Davison (1973), Corfmat and Morse (1976) and Vidyasagar and Viswanadham (1982). All these results are related to the existence of fixed modes.

Definition 1

Let the state space representation of $G(s)$ be given by

$$\dot{x} = Ax + \sum_{i=1}^{l} B_i e_i$$
$$y_i = C_i x, \quad i = 1, 2, \ldots, l \tag{2.7}$$

and suppose we have constant decentralized output feedback of the form

$$u_i = -K_i y_i, \quad i = 1, 2, \ldots, l \tag{2.8}$$

to yield a closed-loop system

$$\dot{x} = (A - BKC)x \tag{2.9}$$

where

$$K = \text{block diag}\,[K_1 \quad K_2 \quad \ldots \quad K_l], \quad B = [B_1 | B_2 | \ldots | B_l] \quad \text{and} \quad C' = [C_1' | C_2' | \ldots | C_l']$$

Then the *fixed modes of* (2.7) *under decentralized feedback* are the zeros of the fixed polynomial $\alpha(s)$ defined by

$$\alpha(s) \triangleq \underset{K_i}{\text{gcd}} \det [sI - A + BKC] \tag{2.10}$$

where gcd denotes the greatest common divisor of det $[sI - A + BKC]$ for block diagonal K, K_i taking on all real matrix values.

An alternative characterization of decentralized fixed modes in terms of certain minors of $G(s)$ is presented elsewhere (Vidyasagar and Viswanadham 1982). In further analysis, we assume that $\alpha(s)$ in (2.10) is at least a stable polynomial. In the more specific case where **D**-Mode compensators are required, $\alpha(s)$ must have its zeros in **D**.

One possible approach towards constructing decentralized controllers is to use constant output feedback on $l-1$ channels,

$$e_i = \bar{u}_i - K_i y_i, \quad i = 1, 2, \ldots, l-1$$

and to make all of the unfixed modes controllable and observable through channel l.

Theorem 1 below gives the conditions under which this can be done. Then one could use standard multivariable feedback design methods to determine the control law that stabilizes the entire system using the input/output pair (u_l, y_l) (Corfmat and Morse 1976, Vidyasagar and Viswanadham 1982 a); this approach is summarized below.

The above design procedure of using constant output feedback on all but one channel can be followed only for *strongly connected systems*. Let I be any subset of the integers $L = \{1, 2, \ldots, l\}$, and $L - I$ be the complement of I in L. Define

$$G_{I,L-I} \triangleq [G_{ij}]_{i\in I, j\in L-I}$$

Then $G_{I,L-I}$ is called a complementary subsystem of $G(s)$, and $G(s)$ is strongly interconnected if and only if all the complementary subsystems are non-zero. Non-strongly connected systems could be treated, first, by decomposing the given system into strongly connected subsystems, and then by compensating each strongly connected subsystem by the above design procedure. For further analysis, we assume that $G(s)$ satisfies this assumption. Non-strongly connected systems can easily be treated (Corfmat and Morse 1976, Vidyasagar and Viswanadham 1982 a), so this assumption is made for convenience only.

Suppose now we apply constant output feedback on the first $(l-1)$ channels

$$e_i = -K_i y_i + \bar{u}_i, \quad i = 1, 2, \ldots, (l-1) \tag{2.11}$$

and let $Q_{l,l}$ be the l, l block of the closed-loop transfer function matrix

$$Q = G[I + \tilde{K}G]^{-1} \tag{2.12}$$

where $\tilde{K} = \text{block diag}\,[K_1 \quad K_2 \quad \ldots \quad K_{l-1} \quad 0]$. We are interested in relating the characteristic polynomial of $Q_{l,l}$ to that of G. The following theorem has been established (Corfmat and Morse 1976, Vidyasagar and Viswanadham 1982 a).

Theorem 1

Let $\alpha(s)$ be the fixed polynomial of $G(s)$ under decentralized feedback, as defined in (2.10). Define $\tilde{K} = \text{block diag}\,[K_1 \quad K_2 \quad \ldots \quad K_{l-1} \quad 0]$ and let $Q_{l,l}$ be the l, l block of the matrix $Q = G[I + \tilde{K}G]^{-1}$. Suppose G is strongly connected and has no zero rows or columns. Then for almost all $K_1, \ldots, K_{l-1}$, the characteristic polynomial of $Q_{l,l}$ is given by

$$Q_{l,l} = \det\,[I + \tilde{K}G] \cdot \phi/\alpha$$

where ϕ is the characteristic polynomial of G. In addition, if $\alpha = 1$, then $Q_{l,l}$ is minimal for almost all $K_1, K_2, \ldots, K_{l-1}$.

It is clear from Theorem 1 that if we apply decentralized output feedback, $e_i = -K_i y_i + \bar{u}_i$, $i = 1, 2, \ldots, (l-1)$, then for almost all $K_1, \ldots, K_{l-1}$ the entire large system is controllable and observable from control station l, i.e. through the input–output pair (e_l, y_l) if $\alpha = 1$. Theorem 1 remains valid even under dynamic compensation. Suppose we apply dynamic output feedback on the first $(l-1)$ channels

$$e_i = -C_i(s)y_i(s) + \bar{u}_i(s), \quad i = 1, 2, \ldots, (l-1) \tag{2.13}$$

and let $\bar{Q}_{l,l}(s)$ be the l, l block of the closed-loop transfer function matrix

$$\bar{Q}(s) = G[I + \tilde{C}G]^{-1} \tag{2.14}$$

where $\tilde{C} = \text{block diag}\,[C_1 \quad C_2 \quad \ldots \quad C_{l-1} \quad 0]$. Suppose $\beta_j(s)$ is the characteristic polynomial of $C_j(s)$. Then from previous results (Wang and Davison 1973, Corfmat and Morse 1976, Vidyasagar and Viswanadham 1982), using Theorem 1 and the fact that fixed modes are invariant under dynamic feedback, we obtain the following.

Corollary 1

Let $\alpha(s)$ be the fixed polynomial of $G(s)$ under decentralized feedback as defined in (2.10) and $\bar{C}$, $\bar{Q}$ and $\bar{Q}_{l,l}$ be as defined above. Let $G(s)$ be strongly connected. Then for almost all $C_1, C_2, \ldots, C_{l-1}$ the characteristic polynomial of $\bar{Q}_{l,l}$ is given by

$$\bar{Q}_{l,l} = \det[I + \bar{C}G] \cdot \phi \cdot \beta_1 \ldots \beta_{l-1}/\alpha$$

If $\alpha = 1$, then $\bar{Q}_{l,l}$ is minimal with this characteristic polynomial for almost all $C_1, C_2, \ldots, C_{l-1}$.

The above theorem and corollary state that it does not matter how C_1, $C_2, \ldots, C_{l-1}$ are chosen. As long as $\bar{Q}_{l,l}(s)$ is stabilizable and detectable, then $G(s)$ is decentrally stabilizable by choice of a suitable C_l. Here we choose the controllers $C_1, \ldots, C_l$ sequentially so that $\bar{Q}_{k,k}$, $k = 1, 2, \ldots$ is stabilized at each step. One benefit of this design procedure is that if the lth station fails (the sensor or actuator becomes open), then the failed system will remain stable.

2.2. *Some results on feedback stabilization*

In this section, we briefly present certain recent results on feedback stabilization (Desoer *et al.* 1980). As mentioned before, the term 'stabilization' should be understood in the extended sense (Fig. 2). Consider the feedback system shown in Fig. 3. Let $G(s)$ be a $q \times m$ rational transfer function matrix and $C(s)$ be an $m \times q$ compensator matrix. Let $\det[I + GC] \neq 0$.

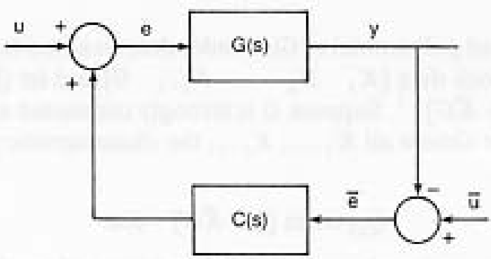

Figure 3. General feedback system configuration.

From Fig. 3 we can easily write down the following equation:

$$\begin{bmatrix} \bar{e} \\ e \end{bmatrix} = \begin{bmatrix} \bar{u} \\ u \end{bmatrix} - \begin{bmatrix} 0 & G \\ -C & 0 \end{bmatrix} \begin{bmatrix} \bar{e} \\ e \end{bmatrix} \tag{2.15}$$

One can rewrite the above equation as

$$\begin{bmatrix} \bar{e} \\ e \end{bmatrix} = \begin{bmatrix} (I + GC)^{-1} & -G(I + CG)^{-1} \\ C(I + GC)^{-1} & (I + CG)^{-1} \end{bmatrix} \begin{bmatrix} \bar{u} \\ u \end{bmatrix} \tag{2.16}$$

or more concisely

$$\tilde{e} \triangleq H(G, C)\tilde{u} \tag{2.17}$$

We say that (G, C) is a *stable pair* if H is a stable rational matrix. It is necessary and sufficient that all four transfer function submatrices in (2.16) be stable; any three of them could be stable while the fourth is not (Desoer and Chan 1975).

We now present a parametrization of all stabilizing compensators of $G(s)$ using coprime factorizations. First we establish some definitions and notation.

Definition 2

Let **S** denote the set of all rational functions (ratios of polynomials in s with real coefficients) that are bounded at infinity and whose poles have negative real parts, i.e. $\mathbf{S}(s)$ consists of all *proper, stable, rational functions.* Under the usual definitions of addition and multiplication in the field of polynomials in s with real coefficients, the set $\mathbf{S}(s)$ is a commutative ring with identity and is a domain.

Definition 3

Let a function u in **S** be called a *unit* if and only if its reciprocal is also in **S**, i.e. u^{-1} is stable and proper. Units in **S** are invertible minimum-phase transfer functions.

Defintion 4

Let $R^{q\times m}(s)$ denote the set of matrices of rational functions of dimension $q \times m$.

Definition 5

Let $\mathbf{M}(\mathbf{S})$ denote the set of matrices with elements in **S** of whatever dimension. If the dimension of the matrix is important, then we will mention this fact explicitly. Thus, $\mathbf{M}(\mathbf{S})$ denotes the set of stable transfer function matrices of any dimension.

Definition 6

Let a matrix $U \in \mathbf{M}(\mathbf{S})$ be called *unimodular* if its inverse is also in $\mathbf{M}(\mathbf{S})$, i.e. det U is a unit.

Now, we shall use these concepts as the basis for defining *coprime factorization.* Given any scalar rational function h, we can find two functions f and g in **S** such that $h = f/g$ and such that f and g are relatively prime (i.e. 1 is the greatest common divisor of f and g). Such a pair (f, g) is called a coprime factorization of h. It is essential to recognize that we are doing factorizations in the ring **S**, and not in the ring of polynomials. Similarly, given any $G(s) \in R^{q\times m}(s)$, we can find a $q \times m$ matrix $N(s) \in \mathbf{M}(\mathbf{S})$ and an $m \times m$ matrix $D(s) \in \mathbf{M}(\mathbf{S})$ such that $G(s) = N(s)D^{-1}(s)$ and the matrices N, D are right coprime. This, in turn, means that there exist $X(s) \in \mathbf{M}(\mathbf{S})$ and $Y(s) \in \mathbf{M}(\mathbf{S})$ such that

$$X(s)N(s) + Y(s)D(s) = I_m \tag{2.18}$$

In the same way, we can find $\tilde{N}$, $\tilde{D}$, $\tilde{X}$ and $\tilde{Y}$ such that $G(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ and $\tilde{D}$ and $\tilde{N}$ are left corpime, i.e.

$$\tilde{N}(s)\tilde{X}(s) + \tilde{D}(s)\tilde{Y}(s) = I_q \tag{2.19}$$

We refer to (N, D) as the *right coprime factorization* (r.c.f.) of $G(s)$ and $(\tilde{D}, \tilde{N})$ as the *left coprime factorization* (l.c.f.) of $G(s)$. Now we state the results that characterize all compensators that stabilize a given strictly proper plant (Desoer *et al.* 1980).

Theorem 2

Let $G \in R^{q \times m}$ be strictly proper and let (N, D) and $(\tilde{D}, \tilde{N})$ be any r.c.f. and l.c.f. of $G(s)$. Let $C(s) \in R^{m \times q}$ be any proper compensator with (N_c, D_c) as r.c.f. and $(\tilde{D}_c, \tilde{N}_c)$ as l.c.f. Then the following conditions are equivalent.

(i) The pair (G, C) is stable.
(ii) The matrix $\tilde{D}_c D + \tilde{N}_c N$ is unimodular.
(iii) The matrix $\tilde{D} D_c + \tilde{N} N_c$ is unimodular. (2.20)
(iv) $C(s) \in \mathbf{C}$.

Theorem 3

Let $G \in R^{q \times m}(s)$ be strictly proper and let (N, D), $(\tilde{D}, \tilde{N})$ be any r.c.f. and l.c.f. of G. Let X, Y, $\tilde{X}$, $\tilde{Y}$ be such that

$$XN + YD = I_m, \quad \tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I_q \tag{2.21}$$

Then

(i) every C such that (G, C) is a stable pair is proper;
(ii) the class $\mathbf{C}$ of all $C(s)$ such that (G, C) is a stable pair is given by

$$C = (Y - R\tilde{N})^{-1}(X + R\tilde{D}), \quad R \in \mathbf{M}(\mathbf{S}) \tag{2.22 a}$$

or

$$C = (\tilde{X} + DS)(\tilde{Y} - NS)^{-1}, \quad S \in \mathbf{M}(\mathbf{S}) \tag{2.22 b}$$

Equation (2.22) comprises two equivalent formulations of the class $\mathbf{C}$ of all stabilizing compensators.

Using Theorem 3, we can now write down all the closed-loop systems obtainable from a stable control system using C as in (2.22).

Theorem 4

Let $G(s)$ be the plant as in Theorem 3 and let $C(s) \in \mathbf{C}$ be chosen as in (2.22 *a*) then the closed-loop system of (2.16) and (2.17) is

$$H(G, C) = \begin{bmatrix} I - N(X + R\tilde{D}) & -N(Y - R\tilde{N}) \\ D(X + R\tilde{D}) & D(Y - R\tilde{N}) \end{bmatrix} \tag{2.23}$$

The expression for $H(G, C)$ with $C(s) \in \mathbf{C}$ chosen as in (2.22 *b*) can be obtained similarly.

From (2.16), it is clear that the closed-loop characteristic polynomial is

$$\phi_c = \det[\tilde{D}_c D + \tilde{N}_c N] = \det[\tilde{D} D_c + \tilde{N} N_c] \tag{2.24}$$

and it is easy to see that

$$(I + GC)^{-1} = D_c[\tilde{D} D_c + \tilde{N} N_c]^{-1} D = D[\tilde{D}_c D + \tilde{N}_c N]^{-1} \tilde{D}_c \tag{2.25}$$

Thus, from (2.24) and (2.25), we have

$$\det[I + GC] = \phi_c / (\beta \cdot \phi) \tag{2.26}$$

where β and ϕ are the characteristic polynomials of G and C. It is possible to derive the multivariable Nyquist criterion (Callier and Desoer 1982) from (2.26).

2.3. *Characterization of all tracking compensators*

Considering Fig. 3 once again, suppose that $\bar{u}$ is equal to a reference signal r and $u=0$. By definition of the tracking problem, r is a persistent (not asymptotically stable) signal. We denote the l.c.f. of $r(s)$ by

$$r(s) \triangleq \tilde{D}_r^{-1}\tilde{N}_r v \tag{2.27}$$

where v is an arbitrary constant vector. The objective now is to characterize the class $\mathbf{C}_T$ of all controllers $C(s)$ such that

(i) $H(G, C)$ is a stable matrix (closed-loop stability);

(ii) the transfer matrix from v to $\bar{e}$ is stable (for asymptotic tracking of r); and

(iii) both (i) and (ii) continue to hold even if the plant is perturbed (robustness).

The main result for robust tracking is as follows.

Theorem 5 (Francis and Vidyasagar 1983)

Consider the plant $G \in R^{q\times m}$ as in Theorem 3, and let α_r be the largest invariant factor of the matrix $\tilde{D}_r$. Then the robust tracking problem has a solution if and only if N and α_r are coprime. When this holds, the class $\mathbf{C}_T$ of all solutions to the problem is given by

$$C = \frac{1}{\alpha_r}(\tilde{X} + DW)V^{-1} \tag{2.28}$$

where V and W are matrices in $\mathbf{M(S)}$ such that

$$\alpha_r V + NW = \tilde{Y} \tag{2.29}$$

This theorem completes the results needed for the developments presented here.

3. Preliminaries for the sequential design method

We now present several results useful in developing a sequential design method for decentralized control systems. Let us recall the problem definition in § 2. Consider the l-channel large-scale system described by (2.2), i.e.

$$y(s) = G(s)e(s)$$

Our aim is to find the block diagonal feedback controller $C(s)$ so that the closed-loop control system defined by (2.4) and (2.5),

$$e = C(\bar{u} - y) + u$$

has its poles in a desired region $\mathbf{D}$ of the s-plane (Fig. 2). Recall that the vectors $\bar{u}$ and u are the reference input and disturbance vectors, respectively (Fig. 3).

From (2.2), (2.4) and (2.5) we have

$$y(s) = (I + GC)^{-1}GC\bar{u} + (I + GC)^{-1}Gu \tag{3.1}$$

We require (3.1) to be stable. The necessary and sufficient conditions for stability of (3.1) in terms of det $[I + GC]$ are well known (Doyle and Stein 1981). Let n_g and n_c

denote the number of right half plane poles of G and C, respectively. Then the closed-loop system is stable if and only if the number of counterclockwise encirclements of the origin by the locus of det $[I+GC]$ as s travels the standard Nyquist contour equals n_g+n_c. Our aim is to find conditions under which $C(s)$ exists such that det $[I+GC]$ satisfies this stability requirement.

3.1. *Generalized decomposition theorem*

The following is a generalization of Rosenbrock's decomposition theorem (1974) wherein the determinant of the return difference matrix is expressed as a product of l terms, each term representing the determinant of the subsystem return difference matrix. This result forms the basis for the sequential design procedure presented in § 4.

To proceed, we need the following notation. For $k=1,2,\dots,l$, let

$$L_k = \text{block diag}\,[C_1 \quad C_2 \quad \dots \quad C_k \quad 0 \quad 0 \quad \dots \quad 0] \tag{3.2}$$

$$\bar{L}_k = \text{block diag}\,[C_1 \quad C_2 \quad \dots \quad C_k] \tag{3.3}$$

$$C^k = \text{block diag}\,[0 \quad 0 \quad \dots \quad C_k \quad 0 \quad \dots \quad 0] \tag{3.4}$$

$$T_k = [I + L_k G] \tag{3.5}$$

$$P^k = G(s)[I+L_kG]^{-1} = (I+GL_k)^{-1}G, \quad P^0 = G \tag{3.6}$$

$$P^k_{i,j} = (i,j)\text{th block matrix of } P^k.$$

Note that T_1 is the return difference matrix with loop 1 closed. Also $P^1_{2,2}$ is the transfer function matrix between e_2 and y_2 when loop 1 is closed with feedback compensator C_1. Similarly, $P^{k-1}_{k,k}$ is the transfer function matrix between e_k and y_k when loops $1, 2, \dots, (k-1)$ are closed with compensators $C_1, C_2, \dots, C_{k-1}$ and the other loops are open. We also note that $L_l = \bar{L}_l = C$.

Theorem 6

With the above notation

$$\text{(i)}\ \det[I+L_kG] = \prod_{i=1}^{k} \det(I + C_i P^{i-1}_{i,i}) \tag{3.7}$$

$$\text{(ii)}\ P^k_{k+1,k+1} = P^{k-1}_{k+1,k+1} - P^{k-1}_{k+1,k}(I+C_kP^{k-1}_{k,k})^{-1}C_kP^{k-1}_{k,k+1}, \quad k=1,2,\dots,l \tag{3.8}$$

and

$$\text{(iii)}\ [I+CG]^{-1} = \prod_{j=1}^{l} (I + C^j P^{j-1})^{-1} \tag{3.9}$$

Proof

Note that from (3.2) and (3.4) we have $C^k = L_k - L_{k-1}$ which gives us the identity

$$C^kG[I+L_{k-1}G]^{-1}[I+L_{k-1}G] = (L_k - L_{k-1})G \tag{3.10}$$

Now using (3.6) in (3.10) we obtain

$$C^kP^{k-1}[I+L_{k-1}G] = (L_k - L_{k-1})G \tag{3.11}$$

Adding an identity matrix to both sides of (3.11) and rearranging we obtain

$$[I+C^kP^{k-1}][I+L_{k-1}G] = (I+L_kG) \tag{3.12}$$

Now (iii) follows immediately by successive application of (3.12) and noting that $L_l = C$ and $L_1 = C_1$.

Using (3.4), we can easily show that

$$[I + C^k P^{k-1}] = \left[\begin{array}{ccc|c|ccc} I & & & & & & \\ & \ddots & & 0 & & 0 & \\ & & I & & & & \\ \hline C_k P^{k-1}_{k,1} & \cdots & C_k P^{k-1}_{k,k-1} & [I + C_k P^{k-1}_{k,k}] & C_k P^{k-1}_{k,k+1} & \cdots & C_k P^{k-1}_{k,l} \\ \hline & & & & I & & \\ & 0 & & 0 & & \ddots & \\ & & & & & & I \end{array}\right] \tag{3.13}$$

Thus

$$\det [I + C^k P^{k-1}] = \det [I + C_k P^{k-1}_{k,k}] \tag{3.14}$$

From (3.12) and (3.5), it is clear that

$$\det T_k = \det T_{k-1} \cdot \det [I + C_k P^{k-1}_{k,k}] = \prod_{i=1}^{k} \det [I + C_i P^{i-1}_{i,i}]$$

thus proving (i).

To prove (ii) we use (3.15) once again to obtain

$$[I + L_k G]^{-1} = [I + L_{k-1} G]^{-1} [I + C^k P^{k-1}]^{-1} \tag{3.15}$$

Multiplying both sides of (3.15) by G, we obtain

$$P^k = P^{k-1} [I + C^k P^{k-1}]^{-1} \tag{3.16}$$

Now use (3.13) in (3.16) and calculate the $(k+1, k+1)$ block of P^k and obtain (3.9).

□

3.2. *Decentralized stabilization*

The decomposition theorem presented above forms the basis for the sequential design method. Notice that $P^{k-1}_{k,k}$ is the so-called local model (Davison and Ozguner 1982) as seen by the kth controller when loops $1, 2, \ldots, (k-1)$ are closed and $(k+1), \ldots, l$ are open. Equation 3.8 gives a recursive expression for computing the local model. The determinant of the return difference matrix is expressed as the product of the subsystem return difference matrices. Thus, the number of encirclements of the origin by the Nyquist locus of $\det [I + L_k G]$ equals the sum of the encirclements of the origin by the Nyquist loci of $\det [I + C_1 G_{11}]$, $\det [I + C_2 P^1_{22}], \ldots, \det [I + C_k P^{k-1}_{k,k}]$. We have the following result.

Theorem 7

Let n_g and n_c be the number of open-loop unstable poles of G and C. Let n_i denote the number of clockwise encirclements of the Nyquist locus of $\det [I + C_i P^{i-1}_{i,i}]$, $i = 1, 2, \ldots, l$ around the origin. Then the closed-loop system under decentralized feedback is stable if and only if $\sum_{i=1}^{l} n_i = n_g + n_c$.

Although this theorem presents necessary and sufficient conditions for decentralized stabilization, it is unclear what class of systems would satisfy the condition $\sum_{i=1}^{l} n_i = n_g + n_c$. In other words, we would wish to determine when stabilization of the pairs $(P_{i,i}^{i-1}, C_i)$, $i = 1, 2, \ldots, l$, would imply stability of the closed loop system. For example, if

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s-1} & \frac{1}{s+2} \end{bmatrix}$$

then $G(s)$ is not decentrally stabilizable although g_{11} and g_{22} are stable.

It has been shown (Wang and Davison 1973, Corfmat and Morse 1976, Vidyasagar and Viswanadham 1982) that the satisfaction of the condition

$$\sum_{i=1}^{l} n_i = n_g + n_c$$

is related to the existence of fixed modes. Indeed, we shall show now that our sequential design procedure yields a stable closed loop system if and only if the fixed modes under decentralized feedback are stable.

Let $G(s)$ be a strongly connected system with fixed polynomial $\alpha(s)$ as in (2.10) and let $\beta_j(s)$ be the characteristic polynomial of $C_j(s)$. Suppose we apply decentralized feedback on all channels, i.e. all l loops are closed via (2.3). Then (Kwakernaak and Sivan 1972, p. 46) the closed-loop characteristic polynomial ϕ_c is given by $\phi_c = \phi \cdot \beta_1 \ldots \beta_l \cdot \det [I + GC]$. Suppose we assume that we close the loops 1, 2, ..., $(l-1)$ via feedback compensators $C_1, \ldots, C_{l-1}$; then, by Corollary 1, the characteristic polynomial ϕ_l of $P_{l,l}^{l-1}$ is

$$\phi_l = \phi\beta_1 \ldots \beta_{l-1} \det [I + L_{l-1} G]/\alpha \tag{3.18}$$

Let $(D_{l,l}, N_{l,l})$ be the r.c.f. of $P_{l,l}^{l-1}$ and $(\tilde{D}_{cl}, \tilde{N}_{cl})$ be the l.c.f. of C_l. Suppose now we choose C_l such that $(P_{l,l}^{l-1}, C_l)$ is a stable pair, i.e.

$$\tilde{D}_{cl} D_{ll} + \tilde{N}_{cl} N_{ll} = U_l \tag{3.19}$$

where U_l is a unimodular matrix. Then from (2.25), we have

$$\det [I + C_l P_{l,l}^{l-1}] = \frac{\det U_l}{\det D_{ll} \det \tilde{D}_{cl}} \tag{3.20}$$

Note that in writing (3.20) we used the fact that $\det D_{ll}$ is the characteristic polynomial of $P_{l,l}^{l-1}$ and $\det \tilde{D}_{cl}$ is the characteristic polynomial of C_l.

Now using (3.7) and (3.17–3.20), we obtain

$$\phi_c = \phi\beta_1 \ldots \beta_l \det [I + L_{l-1} G] \det [I + C_l P_{l,l}^{l-1}] = \alpha \det U_l \tag{3.21}$$

Thus, we require α to be stable for decentralized stabilization to be possible. Further, $C_1, C_2, \ldots, C_{l-1}$ can be chosen arbitrarily. However, in the following section we choose $C_1, C_2, \ldots, C_{l-1}$ such that $(C_k, P_{k,k}^{k-1})$ are *all* stable pairs, $k = 1, 2, \ldots, (l-1)$. This will enable us to distribute the complexity of the controller by not requiring that C_l alone provide overall stability. We summarize the above results as the following theorem.

Theorem 8

Let $G(s)$ be strongly connected with a stable fixed polynomial under decentralized feedback. Then a choice of $C_1, \ldots, C_l$ such that $(C_k, P_{k,k}^{k-1})$ are all stable pairs for $k = 1, 2, \ldots, l$ yields a stable decentralized closed-loop control system.

3.3. *Non-strongly connected systems*

If the original l-channel system is not strongly connected, then one can carry out a decomposition of G into its strongly connected components (Corfmat and Morse 1976, Vidyasagar 1981, § 4.2). This results in ordering the channel indices in such a way that G becomes block-triangular. Specifically, suppose that p is the number of strongly connected components associated with $G(s)$. Then the index set $\{1, 2, \ldots, l\}$ can be permuted and partitioned into p sets

$$\bar{I}_1 = \{1, 2, \ldots, l_1\}$$

$$\bar{I}_2 = \{l_1 + 1, \ldots, l_1 + l_2\}, \ldots$$

$$\bar{I}_p = \left\{ \sum_{i=1}^{p-1} l_i + 1, \ldots, \sum_{i=1}^{p} l_i \right\}$$

in such a way that G has the form

$$G(s) = \begin{bmatrix} \bar{G}_{11} & 0 & 0 & \ldots & 0 \\ \bar{G}_{21} & \bar{G}_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \bar{G}_{p1} & \bar{G}_{p2} & \ldots & \ldots & \bar{G}_{pp} \end{bmatrix} \tag{3.22}$$

and $\bar{G}_{ii}$, $i = 1, 2, \ldots, p$ are strongly connected and have $\bar{l}_i$ channels.

Now the design proceeds as follows. Suppose we apply decentralized feedback

$$C = \text{block diag}\,[\bar{C}_1 \quad \bar{C}_2 \quad \ldots \quad \bar{C}_p] \tag{3.23}$$

where

$$\bar{C}_i = \text{block diag}\,[C_{11} \quad C_{12} \quad \ldots \quad C_{1l_i}] \tag{3.24}$$

We choose $\bar{C}_i$ such that $(\bar{G}_{ii}, \bar{C}_i)$ is a stable pair for $i = 1, 2, \ldots, p$. Since $\bar{G}_{ii}$ is strongly connected, the design of $\bar{C}_i$ proceeds as in § 3.2 for each $i = 1, 2, \ldots, p$. The stability again follows if the fixed polynomial $\alpha(s)$ is stable. The results are similar to those presented elsewhere (Corfmat and Morse 1976, Vidyasagar and Viswanadham 1982 a).

3.4. *Updating formulae*

To determine the controllers $C_1, C_2, \ldots$ we need to find the transfer function matrices $P_{k,k}^{k-1}$, $k = 2, \ldots, l$ (recall that $P_{11}^0 = G_{11}$). One could use (3.8) for this purpose. Furthermore, it is possible to use (3.8) and obtain expressions for the characteristic polynomial and the minors of $P_{k+1,k+1}^k$ in terms of those at the kth stage. We derive these expressions as follows:

Lemma 1

Consider a two-channel system

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \triangleq Ge \tag{3.25}$$

where y_i and e_i are vectors of dimension q_i and m_i, respectively, $i = 1, 2$. Suppose we now apply dynamic feedback to close the first loop,

$$e_1(s) = -C_1(s)y_1(s) \tag{3.26}$$

to obtain

$$y_2 = [G_{22} - G_{21}(I + C_1 G_{11})^{-1} C_1 G_{12}]e_2 \tag{3.27}$$

Define $\bar{q}_1 \triangleq \{1, 2, \ldots, q_1\}$; $\bar{q}_2 \triangleq \{(q_1+1), \ldots, (q_1+q_2)\}$; $\bar{q} \triangleq \bar{q}_1 \cup \bar{q}_2$ and $\bar{m}_1, \bar{m}_2$ and $\bar{m}$ analogously. If J is a set of integers, let $|J|$ denote the number of integers in J. If $|I| = |J|$, then $g\begin{bmatrix} I \\ J \end{bmatrix}$ denotes the minor of G consisting of elements from rows in I and columns in J. Let ϕ be the characteristic polynomial of G and $p\begin{bmatrix} I \\ J \end{bmatrix} = \phi \cdot g\begin{bmatrix} I \\ J \end{bmatrix}$. Then for all $I_2 \in \bar{q}_2$ and $J_2 \in \bar{m}_2$, we have

$$\bar{Q}_{22}\begin{bmatrix} I_2 \\ J_2 \end{bmatrix}\phi(C_1) = \beta_1 p\begin{bmatrix} I_2 \\ J_2 \end{bmatrix} + \sum_{i=1}^{\min(m_1, q_1)} \sum_{\substack{|I_1|=i \\ I_1 \in \bar{q}_1}} \sum_{\substack{|J_1|=i \\ J_1 \in \bar{m}_1}} p\begin{bmatrix} I_1 \cup I_2 \\ J_1 \cup J_2 \end{bmatrix} \tilde{C}_1\begin{bmatrix} J_1 \\ I_1 \end{bmatrix} \tag{3.28}$$

where

$$\phi(C_1) = \beta_1 \phi + \sum_{i=1}^{\min(m_1, q_1)} \sum_{\substack{|I_1|=i \\ I_1 \in \bar{q}_1}} \sum_{\substack{|J_1|=i \\ J_1 \in \bar{m}_1}} p\begin{bmatrix} I_1 \\ J_1 \end{bmatrix} \tilde{C}_1\begin{bmatrix} J_1 \\ I_1 \end{bmatrix} \tag{3.29}$$

and where β_1 is the characteristic polynomial of C_1, $\tilde{C}_1\begin{bmatrix} I \\ J \end{bmatrix} = \beta_1 C_1\begin{bmatrix} I \\ J \end{bmatrix}$, and $C_1\begin{bmatrix} I \\ J \end{bmatrix}$ is the minor of C_1 with rows in I and columns in J.

Proof

See Vidyasagar and Viswanadham (1982 a).

This lemma may be used to obtain the characteristic polynomial and $P_{k,k}^{k-1}$ at each stage of the sequential design procedure.

4. Sequential design procedure

In this section, we present a design procedure based on the results of § 3. This method consists of choosing $C_k(s)$ such that $(P_{k,k}^{k-1}, C_k)$ is a stable pair (in the extended sense, Fig. 2) for $k = 1, 2, \ldots, l$ using the coprime factorization method described in § 2.2. We assume that $G(s)$ is as suggested in § 3.3.

From Fig. 1 and (2.1), (2.6) we can write down the following equations:

$$\begin{bmatrix} \bar{e}_1 \\ e_1 \\ \bar{e}_2 \\ e_2 \\ \vdots \\ \bar{e}_l \\ e_l \end{bmatrix} = \begin{bmatrix} \bar{u}_1 \\ u_1 \\ \bar{u}_2 \\ u_2 \\ \vdots \\ \bar{u}_l \\ u_l \end{bmatrix} - \begin{bmatrix} 0 & G_{11} & 0 & G_{12} & \cdots & 0 & G_{1l} \\ -C_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & G_{21} & 0 & G_{22} & \cdots & 0 & G_{2l} \\ 0 & 0 & C_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & G_{l1} & 0 & G_{l2} & \cdots & 0 & G_{ll} \\ 0 & 0 & 0 & 0 & \cdots & -C_l & 0 \end{bmatrix} \begin{bmatrix} \bar{e}_1 \\ e_1 \\ \bar{e}_2 \\ e_2 \\ \vdots \\ \bar{e}_l \\ e_l \end{bmatrix} \tag{4.1}$$

From (4.1) we obtain

$$\begin{bmatrix} \bar{e}_1(s) \\ e_1(s) \\ \bar{e}_2(s) \\ e_2(s) \\ \vdots \\ \bar{e}_l(s) \\ e_l(s) \end{bmatrix} = \begin{bmatrix} I & G_{11} & 0 & G_{12} & \cdots & 0 & G_{1l} \\ -C_1 & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & G_{21} & I & G_{22} & \cdots & 0 & G_{2l} \\ 0 & 0 & -C_2 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & G_{l1} & 0 & G_{l2} & \cdots & I & G_{ll} \\ 0 & 0 & 0 & 0 & \cdots & -C_l & I \end{bmatrix} \begin{bmatrix} \bar{u}_1(s) \\ u_1(s) \\ \bar{u}_2(s) \\ u_2(s) \\ \vdots \\ \bar{u}_l(s) \\ u_l(s) \end{bmatrix} \tag{4.2}$$

or

$$\tilde{e} \triangleq H(G, C)\tilde{u} \tag{4.3}$$

From the results of § 2.2, it is clear that $C(s)$ defined in (2.3) stabilizes $G(s)$, i.e. (G, C) is a stable pair if $H(G, C)$ is a stable proper transfer function matrix. We note that, in terms of Fig. 3, one can write a re-arranged version of (4.2) as follows:

$$\tilde{e} = \begin{bmatrix} \bar{e} \\ e \end{bmatrix} = \begin{bmatrix} I & G \\ -C & I \end{bmatrix}^{-1} \begin{bmatrix} \bar{u} \\ u \end{bmatrix} \triangleq H\tilde{u} \tag{4.4}$$

Although (4.4) is more compact, we shall use (4.2) in our further analysis. Our objective is to choose $C_1, \ldots, C_l$ sequentially such that $H(G, C)$ is stable; actually we shall characterize all such compensator classes as $\mathbf{C}_1, \ldots, \mathbf{C}_l$.

4.1. *Sequential design algorithm*

Now we present an algorithm for the sequential design of decentralized control systems. We close the channels one by one starting with 1 and at each stage conduct the design using the coprime factorization approach.

Step 1

Calculate the fixed polynomial $\alpha(s)$ defined by (2.10) under decentralized feedback. If the fixed modes are unstable the algorithm terminates; no solution is possible.

Step 2

Let $P^0_{11} = G_{11}$. Also, let (N_{11}, D_{11}) be the right coprime factorization (r.c.f.) and $(\tilde{D}_{11}, \tilde{N}_{11})$ be the left coprime factorization (l.c.f.) of P^0_{11}. Also let X_{11}, Y_{11}, $\tilde{X}_{11}$, $\tilde{Y}_{11}$ be stable proper matrices such that

$$\left.\begin{aligned} X_{11}N_{11} + Y_{11}D_{11} = I \\ \text{and} \qquad \\ \tilde{N}_{11}\tilde{X}_{11} + \tilde{D}_{11}\tilde{Y}_{11} = I \end{aligned}\right\} \tag{4.5}$$

Then from Theorem 3 we have that the class $\mathbf{C}_1$ of all stabilizing compensators of P^0_{11} is given by

$$C_1 = (Y_{11} - R_{11}\tilde{N}_{11})^{-1}(X_{11} + R_{11}\tilde{D}_{11}), \quad R_{11} \in \mathbf{M(S)} \tag{4.6}$$

or

$$C_1 = (\tilde{X}_{11} + D_{11}S_{11})(\tilde{Y}_{11} - N_{11}S_{11})^{-1}, \quad S_{11} \in \mathbf{M(S)} \tag{4.7}$$

Let

$$H_{11}(G, C) = \begin{bmatrix} I & P^0_{11} \\ -C_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} (I + P^0_{11}C_1)^{-1} & P^0_{11}(I + C_1P^0_{11})^{-1} \\ C_1(I + P^0C_1)^{-1} & (I + C_1P^0_{11})^{-1} \end{bmatrix} \tag{4.8}$$

Choose $C_1(s)$ as in (4.6), so that (4.8) yields

$$H_{11}(G, C) = \begin{bmatrix} I - N_{11}(X_{11} + R_{11}\tilde{D}_{11}) & -N_{11}(Y_{11} - R_{11}\tilde{N}_{11}) \\ D_{11}(X_{11} + R_{11}\tilde{D}_{11}) & D_{11}(Y_{11} - R_{11}\tilde{N}_{11}) \end{bmatrix} \tag{4.9}$$

An alternative expression can be obtained for $H_{11}(G, C)$ using (4.7). We are now guaranteed that (P^0_{11}, C_1) is a stable pair. The free parameter matrix R_{11} in (4.6) or S_{11} in (4.7) can then be chosen so that sensitivity is minimized, robustness is maximized, or the dynamic response of $H_{11}(G, C)$ is satisfactory in any sense.

Step 3

Repeat the controller synthesis procedure of Step 2 for each control channel $k = 2, \ldots, l$, each time choosing C_k such that $(P^{k-1}_{k,k}, C_k)$ is a stable pair, as follows: Use (3.8) to determine $P^{k-1}_{k,k}$,

$$P^{k-1}_{k,k} = P^{k-2}_{k,k} - P^{k-2}_{k,k-1}[I + C_{k-1}P^{k-2}_{k-1,k-1}]^{-1}C_{k-1}P^{k-2}_{k-1,k}$$

Note that $P^{k-1}_{k,k}$ is the transfer function matrix between u_k and y_k when loops $1, 2, \ldots, (k-1)$ are closed. Let (N_{kk}, D_{kk}) be the r.c.f. and $(\tilde{D}_{kk}, \tilde{N}_{kk})$ be the l.c.f. of $P^{k-1}_{k,k}$. Let X_{kk}, Y_{kk}, $\tilde{X}_{kk}$, $\tilde{Y}_{kk}$ be stable proper matrices such that

$$X_{kk}N_{kk} + Y_{kk}D_{kk} = I \tag{4.10}$$

$$\tilde{N}_{kk}\tilde{X}_{kk} + \tilde{D}_{kk}\tilde{Y}_{kk} = I \tag{4.11}$$

Then (Theorem 3) the class $\mathbf{C}_k$ of all stabilizing compensators of $P^{k-1}_{k,k}$ is given by

$$C_k = (Y_{kk} - R_{kk}\tilde{N}_{kk})^{-1}(X_{kk} + R_{kk}\tilde{D}_{kk}), \quad R_{kk} \in \mathbf{M(S)}$$

or

$$C_k = (\tilde{X}_{kk} + D_{kk}S_{kk})(\tilde{Y}_{kk} - N_{kk}S_{kk})^{-1}, \quad S_{kk} \in \mathbf{M}(\mathbf{S})$$

The closed-loop response matrix $H_{kk}(G, C)$ can be written in a form analogous to (4.8). The resultant system now has the property that the overall decentralized closed-loop system is asymptotically stable.

Remark 1

If in the procedure it turns out that $P_{k,k}^{k-1} = 0$ for some k, then it is essential that a stabilizing compensator C_k still be applied in order for the algorithm to proceed.

Remark 2

In the above algorithm, the *local model* $P_{k,k}^{k-1}$ at the kth station incorporates the compensator dynamics of the preceding control system loop closures. We emphasize that the *open loop* transfer function matrices G_{kk} cannot in general be used to construct stabilizing controllers. The following example illustrates this point. Let

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & g_{12} \\ g_{21} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

be a two-station large system where y_1, y_2, u_1, u_2 are all scalars. Suppose

$$g_{12} = \frac{s-1}{s(s-2)}, \quad g_{21} = \frac{s-1}{(s+1)(s+2)}$$

Then G has no decentralized fixed modes. It has a blocking zero at $s = 1$, an unstable pole at $s = 2$, and is strictly proper. Thus, it is not strongly stabilizable (stabilizable by a stable compensator) (Vidyasagar and Viswanadham 1982 b). If we were to design the compensators C_1 and C_2 based on $g_{11} = g_{22} = 0$, then C_1 and C_2 would turn out to be stable compensators, and the closed-loop system would not be stable. It is indeed possible to stabilize this system using our sequential algorithm. In fact, $C_1 = 1$ and C_2 such that $(g_{12} \cdot g_{21}, C_2)$ is a stable pair stabilizes the overall system; C_2 would necessarily be unstable.

Remark 3

In the case when $G(s)$ is stable (in particular, when the interaction matrices $G_{ij}(s)$ are stable), one can show algebraically that $H(G, C)$ is stable whenever $(P_{k,k}^{k-1}, C_k)$, $k = 1, 2, \ldots, l$, are all stable pairs. In this instance, we do not have to appeal to the generalized decomposition theorem for proving the stability of $H(G, C)$. To see this, let $l = 2$. Then from (4.2) we obtain

$$H(G, C) = \begin{bmatrix} I & G_{11} & 0 & G_{12} \\ -C_1 & I & 0 & 0 \\ 0 & G_{21} & I & G_{22} \\ 0 & 0 & -C_2 & I \end{bmatrix}^{-1} \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \tag{4.12}$$

$$= \begin{bmatrix} A^{-1} + A^{-1}DS^{-1}CA^{-1} & -A^{-1}DS^{-1} \\ -S^{-1}CA^{-1} & -S^{-1} \end{bmatrix} \tag{4.13}$$

where $S = B - CA^{-1}D$ is the Schur complement of A. Using (4.12) and (3.8) we obtain

$$S = \begin{bmatrix} I & P_{22}^1 \\ -C_2 & I \end{bmatrix} \tag{4.14}$$

It is clear from (4.12) to (4.14) that A^{-1}, S^{-1}, G_{12} and G_{21} are all stable, so $H(G, C)$ is stable. Extension of this result to the case when $l > 2$ follows similarly.

4.2. *Example*

Let us consider a system with two inputs and two outputs

$$G(s) = \begin{bmatrix} 0 & \dfrac{s-1}{s-2} \\ \dfrac{1}{s} & 0 \end{bmatrix} \tag{4.15}$$

It is easy to check that the system in (4.15) has no fixed modes under decentralized feedback. Also $G_{11} = P_{11}^0 = 0$. As suggested in Remark 1, we choose $C_1 = 1$. Obviously (0, 1) is a stable pair. Then we have

$$P_{11}^1 = -\frac{s-1}{s(s-2)} \tag{4.16}$$

If is easy to see that $n_{22} = -(s-1)/(s+1)^2$ and $d_{22} = s(s-2)/(s+1)^2$ is a coprime factorization of (4.16). Furthermore,

$$\frac{(14s-1)}{(s+1)}\frac{(s-1)}{(s+1)^2} + \frac{(s-9)}{(s+1)}\frac{s(s-2)}{(s+1)^2} = 1 \tag{4.17}$$

Thus

$$x_{22} = \left(\frac{14s-1}{s+1}\right) \quad \text{and} \quad y_{22} = \left(\frac{s-9}{s+1}\right)$$

The class of all stabilizing compensators for (4.16) is then given by

$$C_2 = \left[\frac{(s-9)}{(s+1)} - r_{22}\frac{(s-1)}{(s+1)^2}\right]^{-1}\left[\frac{(14s-1)}{(s+1)} + r_{22}\frac{s(s-2)}{(s+1)^2}\right] \tag{4.18}$$

where $r_{22}(s)$ is any member of $\mathbf{M}(\mathbf{S})$. We again point out that we chose $C_1 = 1$ in this example; we could have chosen any stable transfer function instead. It may also be noted that this example is one which is strongly centrally stabilizable (i.e. stabilizable by a stable compensator having off-diagonal terms), but this is not true for the decentralized case. Finally, we should like to recognize that this example is somewhat contrived; it is presented primarily for illustrative purposes.

5. Decentralized tracking problem

Let us consider the plant in Fig. 1 and assume that the disturbances are zero, i.e. $u = 0$, and that the output at local station y_i has asymptotically to track the reference input $\bar{u}_i$. We assume as in § 2.3 that

$$\bar{u}_i \triangleq \tilde{D}_r^{-1} \tilde{N}_r(s) v_i, \quad i = 1, 2, \ldots, l \tag{5.1}$$

where v_i is a constant vector. We are now required to find decentralized compensators $C_1, C_2, \ldots, C_l$ such that stability and asymptotic tracking occur in a robust way. This problem has been studied in the decentralized control context by Davison (1976) and Viswanadham and Ramakrishna (1981). The following method is more design-oriented and uses the sequential algorithm developed in § 4.1.

5.1. *Sequential algorithm for tracking*

For asymptotic tracking at the ith local station, i.e. $\bar{u}_i(t) - y_i(t) \to 0$ as $t \to \infty$, we require the transfer function between $\bar{e}_i$ and v_i (5.1) to be stable. From (5.1) and Fig. 3,

$$\bar{e}_i = [I + GC]_{ii}^{-1} \tilde{D}_r^{-1} \tilde{N}_r v_i \tag{5.2}$$

where $[I + GC]_{ii}^{-1}$ denotes the appropriate block of $[I + GC]^{-1}$. We must choose $C_1, C_2, \ldots, C_l$ such that the $(C_1, P_{11}^0), (C_2, P_{22}^1), \ldots, (C_l, P_{l,l}^{l-1})$ are all stable pairs and $[I + GC]_{ii}^{-1} \tilde{D}_r^{-1} \tilde{N}_r$ is stable. We do this in the following manner.

Step 1

Let $G_{11} = P_{11}^0$. As in § 4.1 determine $N_{11}, D_{11}, \tilde{N}_{11}, \tilde{D}_{11}, X_{11}, Y_{11}, \tilde{X}_{11}, \tilde{Y}_{11}$ to be the usual stable proper matrices. Let α_r be the largest invariant factor of D_r, and assume that $\alpha_r I$ and N_{11} are coprime. Now choose

$$C_1 = \frac{1}{\alpha_r} (\tilde{X}_{11} + D_{11} W_1) V_1^{-1} \tag{5.3}$$

where $V_1 \in \mathbf{M}(\mathbf{S})$, $W_1 \in \mathbf{M}(\mathbf{S})$ and they satisfy

$$\alpha_r V_1 + N_{11} W_1 = \tilde{Y}_{11} \tag{5.4}$$

With the choice of C_1 as in 5.3, from Theorem 4 it follows that (P_{11}^0, C_1) is a stable pair.

Continue Steps 2, 3, ..., k as follows; terminate when $k = l$.

Step k

Assume that N_{kk} and $\alpha_r I$ are coprime pairs. Determine $P_{k,k}^{k-1}$ as in Step 4 of the design algorithm of § 4.1 using (3.8). Now choose C_k as

$$C_k = \frac{1}{\alpha_r} (\tilde{X}_{kk} + D_{kk} W_k) V_k^{-1} \tag{5.5}$$

where $V_k \in \mathbf{M}(\mathbf{S})$, $W_k \in \mathbf{M}(\mathbf{S})$ and they satisfy

$$\alpha_r V_k + N_{kk} W_k = \tilde{Y}_{kk} \tag{5.6}$$

At each step, Theorem 4 ensures that (P_{kk}^{k-1}, C_k) is stable for $k = 2, 3, \ldots, l$.

Now we establish the following result.

Lemma 2

The compensator C_k specified in (5.5) satisfies

$$[I + P_{k,k}^{k-1} C_k]^{-1} = \alpha_r V_k U_k^{-1} \tilde{D}_{kk} \tag{5.7}$$

where U_k^{-1} is a unimodular matrix.

Proof

This follows by direct substitution.

The significance of (5.7) is that α_r is a factor of each entry of $[I + P_{k,k}^{k-1} C_k]^{-1}$. Now we prove the main result of this section.

Theorem 9

Consider the plant $G(s)$ as in § 4.1. Let $(N_{kk}, \alpha_r I)$ be coprime pairs, $k = 1, 2, \ldots, l$ where α_r is the largest invariant factor of $\tilde{D}_r$ in (5.1). Let $\alpha(s)$, the fixed polynomial of the closed-loop system with decentralized feedback, be stable. Then a sequential choice of C_k, $k = 1, 2, \ldots, l$, as (5.5) and (5.6) yields a closed-loop system that is stable and each of its local outputs asymptotically tracks the given reference inputs.

Proof

At each step in the algorithm we are guaranteed that $(P_{k,k}^{k-1}, C_k)$, $k = 1, 2, \ldots, l$, is stable. Since the fixed polynomial is stable, it follows that from Theorem 8 that the closed-loop system is stable.

To prove that asymptotic tracking occurs, we need only show that $(I + GC)^{-1} \tilde{D}_r^{-1}$ is a stable matrix. Then from (5.2) it immediately follows that $\bar{e}(t) \to 0$ as $t \to \infty$. To prove this, we show that α_r, the largest invariant factor of $\tilde{D}_r$, is a factor of every element of $[I + GC]^{-1}$, i.e. we show that

$$[I + GC]^{-1} = \Gamma(s) \alpha_r(s) \tag{5.8}$$

where $\Gamma(s)$ is a stable matrix. From (5.7) and the observation that $\tilde{D}_r^{-1}(s)\alpha_r(s)$ is a stable matrix, it follows that $[I + GC]^{-1} \tilde{D}_r^{-1}$ is a stable matrix. To prove the Theorem, we thus need only establish (5.8). Following the same lines as in Theorem 6 (see (3.12)), it is easy to see that for $k = 1, 2, \ldots, l$

$$[I + GL_k]^{-1} = [I + P^{k-1} C^k]^{-1} [I + GL_{k-1}]^{-1} \tag{5.9}$$

where

$$P^{k-1} = [I + GL_{k-1}]^{-1} G \tag{5.10}$$

Also notice that for $k = 1$

$$[I + GL_1]^{-1} = \begin{bmatrix} (I + G_{11} C_1)^{-1} & 0 \\ X & I \end{bmatrix} \tag{5.11}$$

where X is an arbitrary matrix. Using Lemma 2, it is clear that α_r is a factor of the first block row of $(I + GL_1)^{-1}$. We proceed by induction. Assume that α_r is a factor of the first $(k-1)$ block rows of $(I + GL_{k-1})^{-1}$; we then show that α_r is a factor of the first k block rows of $(I + GL_k)^{-1}$. Suppose we partition G and L as follows:

$$G = \begin{bmatrix} G_{11}^{k-1} & G_{12}^{k-1} & G_{13}^{k-1} \\ G_{21}^{k-1} & G_{22}^{k-1} & G_{23}^{k-1} \\ G_{31}^{k-1} & G_{32}^{k-1} & G_{33}^{k-1} \end{bmatrix} \tag{5.12}$$

where

$$G_{11}^{k-1} = \begin{bmatrix} G_{11} & \cdots & G_{1,k-1} \\ G_{21} & \cdots & G_{2,k-1} \\ \vdots & & \vdots \\ G_{k-1,1} & \cdots & G_{k-1,k-1} \end{bmatrix}, \quad G_{22}^{k-1} = G_{kk} \tag{5.13}$$

and the other block components of G are obviously defined from (5.12) and (5.11). Partition L_{k-1} in the corresponding fashion,

$$L_{k-1} = \begin{bmatrix} \bar{L}_{k-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.14}$$

where

$$\bar{L}_{k-1} = \text{block diag}\,[C_1 \quad C_2 \quad \ldots \quad C_{k-1}] \tag{5.15}$$

With this partition, we obtain

$$[I + GL_{k-1}]^{-1} = \begin{bmatrix} (I + G_{11}^{k-1}\bar{L}_{k-1})^{-1} & 0 & 0 \\ -G_{21}^{k-1}\bar{L}_{k-1}(I + G_{11}^{k-1}\bar{L}_{k-1})^{-1} & I & 0 \\ -G_{31}^{k-1}\bar{L}_{k-1}(I + G_{11}^{k-1}\bar{L}_{k-1})^{-1} & 0 & I \end{bmatrix} \tag{5.16}$$

Using (5.14) and (5.9) we obtain

$$[I + P^{k-1}C^k]^{-1} = \begin{bmatrix} I & (I + G_{11}^{k-1}\bar{L}_{k-1})^{-1}G_{12}^{k-1}C_k(I + P_{kk}^{k-1}C_k)^{-1} & 0 \\ 0 & (I + P_{kk}^{k-1}C_k)^{-1} & 0 \\ 0 & X & I \end{bmatrix} \tag{5.17}$$

where X again denotes an arbitrary matrix. From (5.8), (5.15) and (5.16), one can obtain $(I + GL_k)^{-1}$. Notice that since α_r is a factor of each element of $(I + G_{11}^{k-1}\bar{L}_{k-1})^{-1}$ and $(I + P_{kk}^{k-1}C_k)^{-1}$, it follows that it is a factor of the first $(k+1)$ block rows of $[I + GL_k]^{-1}$. Hence the theorem. □

Theorem 9 requires that $\alpha_r I$ and N_{kk}, $k = 1, 2, \ldots, l$, be coprime for a solution to the tracking problem to exist; it would be interesting to establish a connection between this condition and previous results (Davison 1976, Viswanadham and Ramakrishna 1981). The robustness of the control system follows as in Doyle and Stein (1981). The disturbance rejection problem can be similarly formulated and solved.

6. Conclusions

We have presented a sequential design method for solving the decentralized stabilization and tracking problems. At each stage we characterize the class of all the stabilizing/tracking compensators, **C** and $\mathbf{C}_T$ respectively. We have also established a generalized decomposition theorem (Theorem 6) that provides the proof of stability for our design method and presented a characterization of all possible closed-loop transfer functions $H(G, C)$. This design procedure has the following advantages.

6.1. *Automatic design to basic performance specifications.* At each step, the designer obtains an explicit characterization of the class **C** of all compensators for the loop being closed that results in closed-loop system poles being in a specified closed region **D** of the s-plane, instead of merely stabilizing the closed-loop system. Since one of the primary goals of control system design is to satisfy performance requirements that are often directly related to closed-loop pole location (bandwidth, percentage overshoot, rise time, settling time, etc.), this approach immediately allows the designer to focus on other concerns such as robustness, sensitivity and reliability. Thus, the designer has a clearly-defined, limited optimization problem to solve without concern for loss of basic performance.

6.2. *Simplicity of design.* Other decentralized control schemes (Wang and Davison 1973, Corfmat and Morse 1976, Vidyasagar and Viswanadham 1982, Davison 1976, Viswanadham and Ramakrishna 1981) use constant output feedback on all channels except one and an observer-based dynamic controller on the remaining channel. This dynamic controller is supposed to estimate the entire state of the large system for the purpose of implementing the state feedback law that stabilizes it. Although this control structure is decentralized, it has the disadvantage that all complexity is concentrated in one channel. The sequential approach suggested here tends to distribute the complexity among all the channels. Furthermore, it also clearly indicates the degrees of freedom available in each controller after the overall system stability constraint is met. This freedom could be exploited for robustness improvement, sensitivity minimization, etc.

6.3. *Expandability.* Suppose we expand the system by adding a new subsystem. Then our design procedure will have one more step which involves the characterization of the class $\mathbf{C}_{l+1}$ of all compensators such that $(P^l_{l+1,l+1}, C_{l+1})$ is stable. Note that $C_1, C_2, \ldots, C_l$ need not be altered or retuned. Of course, it is necessary that the expanded system has no unstable fixed modes under decentralized feedback.

6.4. *Use of local models.* The design of $C_1, C_2, \ldots, C_l$ is based on the local models $P^{k-1}_{k,k}$, $k = 1, 2, \ldots, l$, respectively. Although we assumed above that each controller is based on explicit knowledge about G, one could obtain these local transfer functions using on-line identification techniques on a loop-by-loop basis.

6.5. *Limited fault tolerance.* A common failure mode in multivariable control systems is the 'opening' of a loop, in the sense that a sensor, controller, or actuator may fail to transmit the appropriate signal. This condition may be represented by setting the appropriate transfer function to zero. This topic is the main thrust of the work by Viswanadham (1984). In the present case, the degree of fault tolerance is quite modest. If the designer has a concern for reliability in the sense of a particular loop becoming open, then reliability with respect to that one failure may be ensured by designing the compensator for that loop at the last step. This procedure results in a control system in which all of the poles that can be influenced by the remaining live controllers will remain in the region **D** (the only poles that may not lie in **D** are those that correspond to fixed modes with respect to the operating control loops). It is even possible to order the loop closures so that the loop that is second most likely to fail may be designed next-to-last, etc. thus extending this class of fault tolerance to handle failures that may be causally inter-related.

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