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Describing Function Method for Limit Cycle Analysis of Highly Nonlinear Systems

Abstract. Using sinusoidal-input describing functions (SIDF's) is a well-known approach for studying nonlinear oscillations in systems with one dominant nonlinearity [1,2]. In recent years, a number of extensions of the SIDF method have been developed to permit the analysis of systems containing more than one nonlinearity. In many cases, the nonlinear system models that can be treated by such extensions have been quite restricted (limited to a few nonlinearities, or to certain specific configurations; cf. [1]). Furthermore, some results involved only conservative conditions for limit cycle avoidance, rather than actual limit cycle conditions. The technique described in this paper removes all constraints: Systems described by a general state vector differential equation, with any number of nonlinearities, may be analyzed. In addition, the nonlinearities may be multi-input, and bias effects can be treated with little added difficulty.

The general SIDF approach was first fully developed and applied in [3], where a ninth-order highly nonlinear aircraft model was studied. A special case has also been applied to determine limit cycle conditions for rail vehicles; references for this work may be found in [4]. Its power and use are illustrated here by applying it to high-order scalar differential equations with multiple nonlinearities.

1. Introduction - Outline of the General DF Method

The basic idea of the describing function (DF) approach for studying nonlinear system behavior is to replace each system nonlinearity with linear terms whose "gains" are functions of "input amplitudes", where the type of input signal is assumed in advance; this concept is dealt with very thoroughly in [1,2]. In this paper, two cases are considered:

Sinusoidal-Input Describing Functions (SIDF's)

$$f(a \sin \omega t) \approx n(a) \cdot a \sin \omega t \tag{1}$$

Dual-Input Describing Functions (DIDF's)

$$f(b + a \sin \omega t) \approx f_o(a,b) + n(a,b) \cdot a \sin \omega t \tag{2}$$

The DF elements n or f_o, n are mathematically formulated to minimize the approximation error in (1) and (2). For sinusoidal or dual inputs, this is accomplished by retaining the first two terms of the Fourier expansion of $f(b + a \sin \omega t)$. The usefulness of the DF method lies in the subsequent treatment of the resulting quasi-linear model using linear system analytic techniques, which are well established and usually very straightforward to apply. The power of the DF methods is derived from the amplitude-dependence of the DF elements, which accounts for one of the basic effects of nonlinearity. Standard linearization (small-signal or Taylor series linearization) fails to capture this essential property of nonlinear phenomena. A wide variety of SIDF's and DIDF's are catalogued in [1,2],

IX International Conference on Nonlinear Oscillations - Kiev, USSR, Sept. 1981

so we will not consider that aspect of DF theory further.

The most general limit cycle conditions based on DF theory were formulated in [3], and illustrated in [4,5,6]. In summary, we are given

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u}) \tag{3}$$

when $\underline{x}$ is an n-dimensional state vector and $\underline{u}$ is an m-dimensional input vector. It is desired to determine if (3) may exhibit limit cycle behavior when $\underline{u}$ is a vector of constants, denoted $\underline{u}_o$. If the state variables are nearly sinusoidal,

$$\underline{x} \approx \underline{x}_c + \mathrm{Re}[\underline{a}\ \exp\ (j\omega t)] \tag{4}$$

where $\underline{a}$ is a complex amplitude vector and $\underline{x}_c$ is the state vector center value (which is generally not a singularity, or solution to $\underline{f}(\underline{x}_o,\underline{u}_o) = \underline{0}$), then we again neglect higher harmonics, to make the approximation

$$\underline{f}(\underline{x},\underline{u}_o) \approx \underline{f}_{DF}(\underline{x}_c,\underline{a},\underline{u}_o) + \mathrm{Re}[F_{DF}(\underline{x}_c,\underline{a},\underline{u}_o)\ \underline{a}\ \exp\ (j\omega t)] \tag{5}$$

The real vector $\underline{f}_{DF}$ and the matrix F_{DF} are obtained by taking the Fourier expansions of the elements of $\underline{f}(\underline{x}_c + \mathrm{Re}\ \underline{a}\ \exp\ (j\omega t),\underline{u}_o)$, as illustrated below. The assumed limit cycle exists for $\underline{u} = \underline{u}_o$ if $\underline{x}_c$ and $\underline{a}$ can be found so that

(i) $\underline{f}_{DF}(\underline{x}_c,\underline{a},\underline{u}_o) = \underline{0}$

(ii) $[j\omega I - F_{DF}(\underline{x}_c,\underline{a},\underline{u}_o)]\underline{a} = 0,\ \underline{a} \neq \underline{0}$ (6)

(F_{DF} has a pair of pure imaginary eigenvalues, and $\underline{a}$ is the corresponding eigenvector.)

These nonlinear algebraic relations (6) are often called the conditions of harmonic balance; they are generally difficult to solve. Simple examples where hand solution is possible may be found in [4,5], and below. An iterative method, based on successive approximation, can be used successfully for more complicated problems such as the highly complex aircraft performance analysis problem (9 state variables) described in [5].

2. General Application to High-Order Scalar Differential Equations

A special case to which the general result of Section 1 can be easily applied is given by the n^{th} order scalar differential equation

$$f(x,\ Dx,\ D^2x,\cdots,D^nx,u_o) = 0 \tag{7}$$

where x is a scalar variable, u_o is a single constant input, and D^k represents the differential operator d^k/dt^k. The nonlinear differential equation is quasi-linearized to obtain the following harmonic balance conditions:

(i) $f_o(x_c,a,\omega,u_o) = 0$ (8)

(ii) Two roots of the quasi-linear characteristic equation

$$n_n s^n + \cdots + n_1 s + n_o = 0,\ n_i = n_i(x_c, a, \omega, u_o),\ i = 0,1,\cdots,n \quad (9)$$

must be pure imaginary, where f_o and n_i are the DIDF's for $f(\cdot)$. The conditions (8, 9) are derived from (6) by choosing the state vector $\underline{x} = [x, Dx, \cdots D^{n-1}x]^T$; a great simplification in (ii) occurs because by inspection

$$\underline{x}_c = [x_c, 0, 0, \cdots, 0]^T,\ \underline{a} = [a, j\omega a, -\omega^2 a, \cdots, (j\omega)^{n-1} a]^T \quad (10)$$

For a limit cycle to be predicted in the system (7), one must be able to find (x_c, a, ω) — three real unknown values -- so that (8, 9) holds. The describing functions f_o and n_i, = 0,1,$\cdots$,n, are often rather readily obtained by assuming that $\underline{x}$ is of the form (4), with $\underline{x}_c$ and $\underline{a}$ given in (10), and performing a Fourier series expansion of the scalar nonlinearity $f(\cdot)$. For single-input nonlinear terms in (7), one can refer directly to [1,2]; multiple-input nonlinearities, e.g., $x\dot{x}$ and $x \operatorname{sgn} \dot{x}$, are also quite tractible if they are of the power-law type ($x\dot{x}$, $x^2\dot{x}$, etc.) as shown in the following illustration.

3. Illustration

Given the scalar differential equation

$$D^3x + D^2x + 2(1+kx^2)Dx + 3(1+x^2)x = u_o \quad (11)$$

we quasi-linearize to obtain the harmonic balance relations as follows:

$$x^3 \approx x_c(x_c^2 + \tfrac{3}{2}a^2) + 3(x_c^2 + \tfrac{1}{4}a^2) \cdot a \cos \omega t$$

$$x^2 Dx \approx 0 + 0 \cdot a \cos \omega t + (x_c^2 + \tfrac{1}{4}a^2) \cdot (-a\omega \sin \omega t)$$

(which are simply obtained using trigonometric identities) yields

$$f_o = 3x_c\ (1 + x_c^2 + \tfrac{3}{2}a^2) = u_o \quad (12)$$

$$s^3 + s^2 + 2[1 + k(x_c^2 + \tfrac{1}{4}a^2)]s + 3[1 + 3(x_c^2 + \tfrac{1}{4}a^2)] = 0 \quad (13)$$

which may be denoted succinctly as $s^3 + s^2 + \beta s + \gamma = 0$. Using any method of linear system analysis, the characteristic equation has imaginary roots if

$$\Delta \triangleq \beta - \gamma = (2k - 9)(x_c^2 + \tfrac{1}{4}a^2) - 1 = 0 \quad (14)$$

Seeking solutions (x_c, a) to (12) and (14) reveals a great deal of information about limit cycle conditions in (11). First, limit cycles cannot exist if $k \leq 4.5$, because both left-hand-side terms of (14) are then negative and equality is impossible for any real (x_c, a). To continue, choose k = 6 in (11), so that limit cycles can exist. Then (14) reduces to

$$3\ (x_c^2 + \tfrac{1}{4}a^2) = 1 \quad (15)$$

so, for k = 6, $|x_c|$ cannot exceed $1/\sqrt{3}$ - otherwise a is not real in (15). Substituting from (15) to eliminate a from (12) yields

$$3x_c(3 - 5x_c^2) = u_o \tag{16}$$

Thus, no limit cycles exist for $|u_o| > 6/\sqrt{5} = 2.68$, one limit cycle exists for $|u_o| < 4/\sqrt{3} = 2.31$, and two limit cycles exist for $2.31 < |u_o| < 2.68$ – the reasons for this statement are depicted graphically in Fig. 1. * For $u_o = 2.4$, two limit cycles exist; solving (15), (16) leads to the solutions summarized in Table 1.

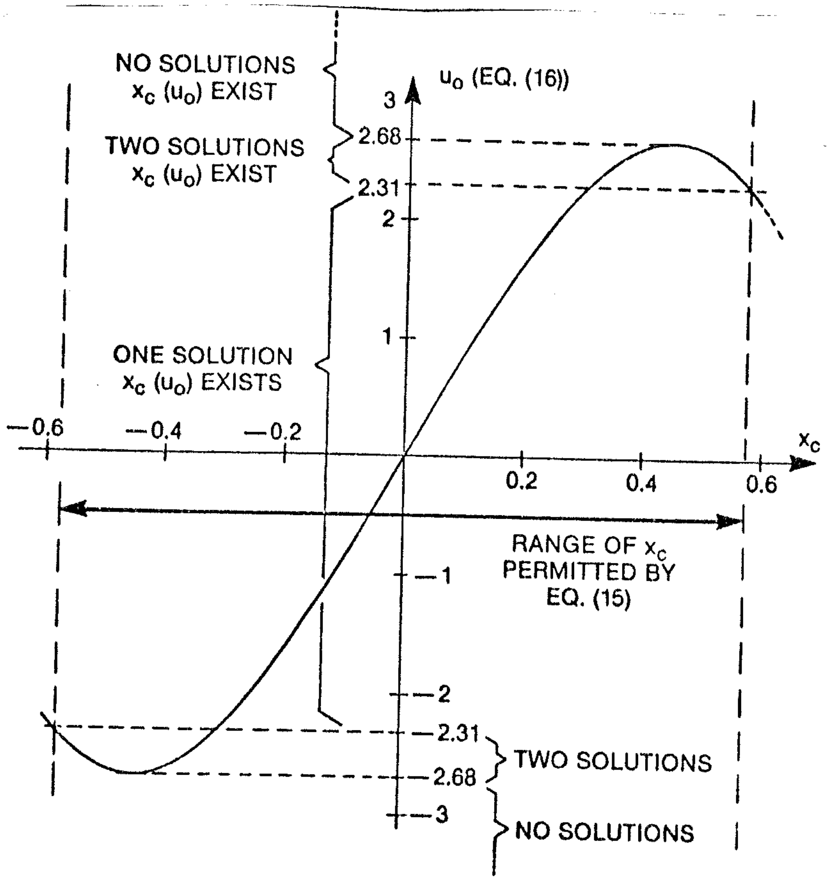

Fig. 1 Solving for Limit Cycle Conditions

Table 1

Limit Cycle Conditions for $k = 6$, $u_o = 2.4$

	x_c	a	ω
LC1	0.323	0.958	2.45
LC2	0.561	0.272	2.45

It is readily shown that LC1 is stable, LC2 is inside LC1 and is unstable, and a stable equilibrium at $x_o = 0.592$ is inside LC2. These conditions may be proven using conventional linearization methods near x_o and the following proposed limit cycle stability/instability criterion:

Proposition: A limit cycle with parameters (a, x_c) is stable [unstable] if for oscillation amplitude $(a + \delta a)$ the quasi-linear characteristic equation has all roots in the left-half plane (LHP) for $\delta a > 0$ [$\delta a < 0$], and two complex conjugate roots in the right half plane (RHP) for $\delta a < 0$ [$\delta a > 0$] (all other roots being in the LHP).

To apply this criterion, we must first solve (12) to obtain δx_c in terms of δa: for constant u_o,

$$\frac{\partial u_o}{\partial x_c}\delta x_c + \frac{\partial u_o}{\partial a}\delta a = 0 \rightarrow \delta x_c = -\frac{3ax_c}{1 + 3x_c^2 + \frac{3}{2}a^2}\delta a \tag{17}$$

* It is important to note that the bifurcation from one to two limit cycles occurs for those values of u_o (± 2.31) for which conventional linearization indicates that the equilibrium changes from unstable to stable.

Then, returning to (13) and (14), one can apply any linear system stability method (e.g., Routh-Hurwitz) to show that $\delta\Delta > 0$ for all poles in the LHP, and $\delta\Delta < 0$ for two RHP complex conjugate roots. Inspect $\delta\Delta$: from (14) and (17),

$$\delta\Delta = \frac{\partial\Delta}{\partial x_c}\delta x_c + \frac{\partial\Delta}{\partial a}\delta a = \frac{3a(2+3a^2-18x_c^2)}{2(2+3a^2+6x_c^2)}\delta a$$

For LC1 substitution from Table 1 reveals that sign $(\delta\Delta)$ = sign(δa), so $\delta a > 0 \Rightarrow \delta\Delta > 0 \Rightarrow$ all poles in the LHP $\Rightarrow$ LC1 is stable, whereas for LC2 the opposite result ensues.

The author has not proven the proposed criterion, but he has never found it to fail. It can be justified heuristically by arguing that solutions near a stable limit cycle must be similar to $(x_c + \delta x_c) + (a + \delta a)\, e^{-\alpha t} \cos \omega t$, where sign$(\alpha)$ = sign (δa), and conversely for unstable limit cycles. It is clear that failing to consider the perturbation in x_c, δx_c, can lead to false stability assessments; in this illustration, for example, if x_c is not perturbed, one obtains $\delta\Delta = \frac{3}{2} a\delta a$ which incorrectly implies that both LC1 and LC2 are stable.

4. Summary and Conclusions

A well-established method now exists for studying limit cycle behavior in complicated nonlinear systems. For higher-order systems (n>2) it is often necessary to perform extensive computer calculations; however, for scalar systems some noteworthy simplifications occur which allow quite difficult-seeming problems to be treated rather easily, as the illustration above shows. The study of a system with two nonlinearities, one of which is two-input, and with oscillations about unknown center values is not possible with earlier DIDF methods [1,2].

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