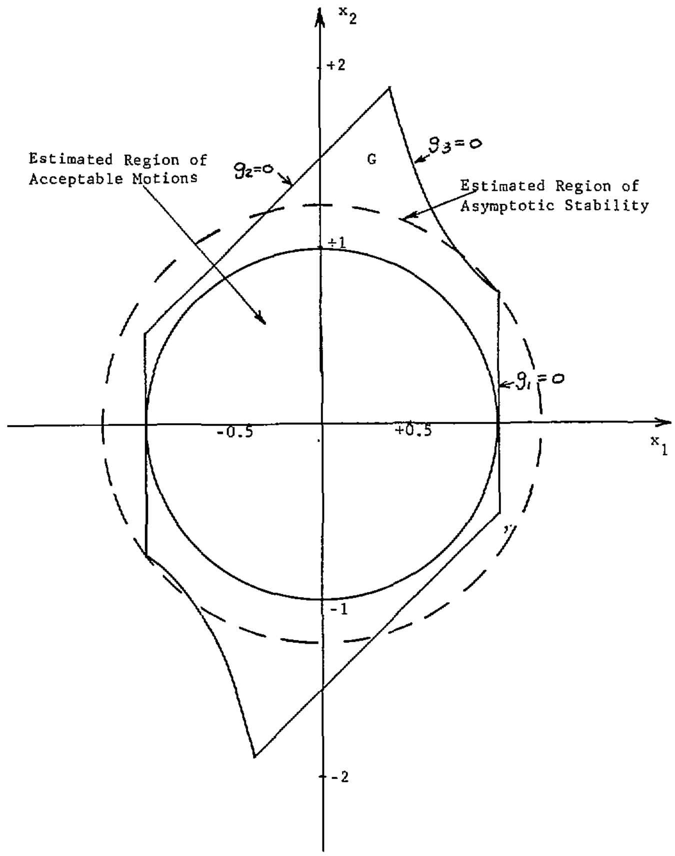

Fig. 2. Estimate of the region of acceptable motions for example problem.

$$J_4 = \min \left\{ V(\mathbf{x}) + \lambda_4 \left[<\frac{\partial V}{\partial \mathbf{x}}, \dot{\mathbf{x}}> \right] \right\} \qquad (12)$$
$$<\frac{\partial V}{\partial \mathbf{x}}, \dot{\mathbf{x}}> = 0.$$

These constrained minimization problems may be solved, in general, by replacing the Lagrange multipliers by penalty coefficients and solving the resulting unconstrained minimization problems via a search technique such as the one described by Fletcher and Powell [6]. This search technique is particularly well suited for the type of function discussed in this correspondence.

For a Liapunov function given by $V(\mathbf{x}) = x_1^2 + x_2^2$, the preceding minimization problems were solved and, using the preceding procedure, an estimate of the region of acceptable motions was determined to be $\Omega_\beta = \{\mathbf{x} \mid V(\mathbf{x}) < \beta\}$, where $\beta = 1.00 = \min [J_1, J_2, J_3, J_4]$. Fig. 2 shows the estimate of the region of asymptotically stable acceptable motions.

Conclusions

The problem of determining or estimating the region of asymptotic stability has been studied by numerous investigators. For practical applications, however, engineers are frequently interested in determining or estimating the set of initial conditions for which the system is not only stable in the sense of Liapunov (Problem A), but also such that the trajectory of the system does not violate a set of constraints (Problem B). The concept of acceptable motions has been introduced to facilitate the study of this problem. The idea of acceptable motions is related to that of practical stability in that a set of acceptable states is specified for the mathematical model. A similar idea was proposed by Hahn [7] to estimate the region of asymptotic stability using linear equations.

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Stability of the Damped Mathieu Equation

In two recent correspondence items [1], [2], the stability of the damped Mathieu equation

$$\ddot{x} + 2\zeta\dot{x} + (a - 2q\cos 2t)x = 0$$

has been treated for small damping ($\zeta \ll 1$). In a recent study [3] of this equation, the authors have been able to improve on the stability boundary as follows.

a) Michael [1] requires $|q| < \frac{1}{2}a$ and $q_- < q < q_+$ where

$$q \pm = \pm \frac{2a\zeta}{(a+1)^2} \{[(2\zeta)^2 + (a+1)^2]^{1/2} - 2\zeta\}$$

which for $\zeta \ll 1$ becomes

$$|q_1| < \frac{2a\zeta}{a+1}.$$

b) Parks [2] (using the circle criterion [4]) requires

$$|q_2| < \zeta\sqrt{a - \zeta^2} \approx \zeta\sqrt{a} \quad \text{for } a \gg \zeta^2.$$

c) The authors [3] (using a theorem of Brockett and Forys [5]) require

$$|q_3| < \sqrt{a(a+1)}\,\zeta \quad \text{for } a \geq 0(1).$$

d) The authors [3] (using a new theorem [6]) require

$$|q_4| < \frac{\pi}{2} a\zeta \quad \text{for } a \gg \zeta^2.$$

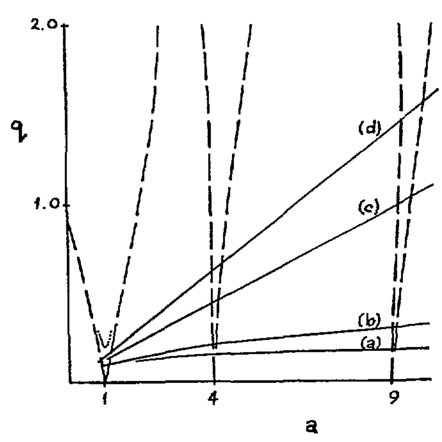

Fig. 1. Dashed lines: exact boundaries, $\zeta = 0.0$; solid lines: boundaries by (a) to (d), $\zeta = 0.1$; dotted line (at $a \approx 1$): exact boundary, $\zeta = 0.1$.

The four stability boundaries are shown for $\zeta = 0.1$ on Fig. 1. The exact boundary at $a = 1$ (as found in [3]) is also shown, for the sake of comparison.

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Remarks by P. C. Parks[1]

The further improvements in explicit criteria by Narendra and Taylor are interesting, but must be applied with caution. Using the recently published results of Smirnov [7], which greatly extend those by McLachlan [8], it is possible to compare these various criteria with the exact stability boundaries. Smirnov gives in his Fig. 1 plots of the iso$-\mu$ curves for the undamped Mathieu equation

$$\ddot{y} + (a' - 2q' \cos 2t)y = 0 \qquad (1)$$

covering $0 \leq a \leq 16$ and $0 \leq q \leq 20$.

The damped Mathieu equation

$$\ddot{x} + 2\zeta\dot{x} + (a - 2q \cos 2t)x = 0 \qquad (2)$$

may be transformed by the substitution

$$x = e^{-\zeta t}y \qquad (3)$$

into

$$\ddot{y} + (a - \zeta^2 - 2q \cos 2t)y = 0 \qquad (4)$$

which is the same form as (1) with $a' = a - \zeta^2$ and $q' = q$. We are then interested in the iso$-\beta$ curves of (4) corresponding to $\beta = \zeta$. For $\beta < \zeta$, the solutions of (2) are damped, since the $e^{-\zeta t}$ term in (3) overcomes the $e^{\beta t}$ term appearing in the solution of (4), which is of the form $A_1 e^{\mu t} f_1(t) + A_2 e^{-\mu t} f_2(t)$ where f_1 and f_2 are periodic functions of t.

Using Smirnov's diagram, it is possible to construct counter examples to Narendra and Taylor's results in q_3 and q_4, unless ζ really is small compared with unity.

It should be pointed out that the result of (2) does not depend on ζ being small. Using a recent result of Infante [9, eq. (3)], a further criterion is

$$|q_5| < \sqrt{2}\sqrt{a}\,\zeta$$

which is an improvement of $\sqrt{2}$ on the circle criterion for large a. This holds also for all $\zeta > 0$.

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Reply by J. H. Taylor[2]

In the absence of Professor Narendra, I would like to stress several points: i) the use of iso-μ curves (McLachlan, Smirnov) can only be made where they exist (e.g., $0 \leq a \leq 16$) and interpolations must be made between curves for values of μ other than those given, whereas the above results are valid for all a $\geq 0(1)$ and require no interpolation; ii) the boundaries in c) and d) are proportional to a rather than $\sqrt{a}$, which is quite significant for larger values of a as is evident in the figure; and iii) the theorems used in c) and d) are valid for larger ζ but the results are not so simply stated. It is felt that by placing so much emphasis on the clearly stated invalidity for large ζ, Parks obscures the main point: that under the constraint $\zeta \ll 1$, the theorems of sections c) and d) yield far less conservative stability boundary estimates. [One might add that they are considerably less conservative than the result cited by Parks using a theorem due to Infante.]

[1] Manuscript received June 7, 1968.
[2] Manuscript received December 20, 1968.

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Positive Real Function and Lyapunov Function Generation

***Abstract*—The invariance of the Lyapunov property for functions under the basic operations of addition, multiplication and, in some cases, division is shown. The results are compared with those known for positive real functions.**

INTRODUCTION

Lyapunov functions in the study of the stability of control systems [8] are as important as are the positive real functions in the study of network realizability theory. Positive real functions have been the subject of intensive research by network theorists, and a considerable amount of literature is available on the generation [1]–[5] of such functions and matrices by elementary operations. The purpose of this correspondence is to show that some of the basic operations are common to the generation of both Lyapunov functions (LFs) and positive real functions (PRFs), while some others can be used to generate either LFs or PRFs, but not both.

Definition: It is well known [6] that a scalar function $V(x_1, x_2, \cdots, x_n)$ is a Lyapunov function if and only if

1) $V(x_1, x_2, \cdots, x_n)$ is continuous together with its first partial derivatives in a certain open region A about the origin;
2) $V(0, 0, \cdots, 0) = 0$;
3) outside the region (and always in A), V is positive; and
4) $dV/dt \leq 0$ in A.

These four conditions will be referred to as the LF requirements, for brevity.

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LYAPUNOV FUNCTION GENERATION

It is known that if Z_1 and Z_2 are PRFs, then $Z_1 + Z_2$ is also a PRF [1]. A similar result valid for Lyapunov functions is written below in the form of a theorem.

Theorem 1: If V_1 and V_2 are two LFs, then $V_1 + V_2$ is also a LF.

The validity of the preceding theorem follows from the fact that all the LF requirements are met by $V_1 + V_2$.

It is known that the product $Z_1 Z_2$ of two PRFs Z_1 and Z_2 is not necessarily a PRF [4]. However, the Lyapunov property is invariant under multiplication, as seen in the next theorem.

Theorem 2: If V_1 and V_2 are two LFs, then $V_1 V_2$ is also an LF

Proof: That $V_1 V_2$ satisfies the first three of the four LF requirements is obvious.

Also,

$$\frac{d}{dt}(V_1 V_2) = V_1 \frac{dV_2}{dt} + V_2 \frac{dV_1}{dt} \leq 0.$$

Therefore, the theorem is proved.

It is known that the quotient of two PRFs Z_1 and Z_2 is a PRF only in certain cases (for example, when Z_1 and Z_2 are both RC realizable)[7]. An analogous result is stated below for LFs.

Theorem 3: If V_1 and V_2 are two LFs, then $V = V_1/(V_2 + \alpha)$, $(\alpha > 0)$ is an LF if

$$\frac{V_1}{V_2 + \alpha} \geq \frac{\dfrac{dV_1}{dt}}{\dfrac{dV_2}{dt}},$$

throughout the open region A under consideration.

Proof: LF requirement 1) is satisfied by V. LF requirements 2) and 3) are satisfied by V because $\alpha > 0$.

Finally,

$$\frac{dV}{dt} = \frac{(V_2 + \alpha)\dfrac{dV_1}{dt} - V_1 \dfrac{dV_2}{dt}}{(V_2 + \alpha)^2} \leq 0$$

provided

$$\frac{V_1}{V_2 + \alpha} \geq \frac{\dfrac{dV_1}{dt}}{\dfrac{dV_2}{dt}}.$$

Therefore, the theorem is proved.

It is true that the reciprocal of a PRF is a PRF [1]. However, the reciprocal $1/V$ of an LF, V, cannot be an LF since $1/V$ violates LF requirements 2) and 4).

CONCLUSION

Possible generation of Lyapunov functions by elementary operations is proved.