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Random-input describing functions for multi-input non-linearities†

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Random-input describing functions for three broad classes of multi-input non-linearities are shown to be readily derived from quasi-linear representations of constituent non-linearities that have fewer input variables.

1. Introduction

Random-input describing function theory plays a central role in a recently developed method for the statistical analysis of non-linear systems with random inputs. The technique represents an extension of linear covariance analysis, based on statistical linearization ; for simplicity, it is designated CADET™—the Covariance Analysis DEscribing function Technique. For details, see Gelb and Warren (1973).

As demonstrated in the above reference, and in Warren *et al.* (1973), Siegel and Warren (1973), and Taylor and Price (1974), the CADET methodology has proven to be a powerful tool for the statistical analysis of highly non-linear system models. In developing realistic models for particular applications, a number of scalar multi-input non-linearities have been encountered having one of the forms

$$\left.\begin{aligned} f_1 &= v_1 g(v_2) \\ f_2 &= v_1 g(v_2, v_3) \\ f_3 &= v_1{}^k g(v_2), \quad k=2, 3, \ldots \end{aligned}\right\} \tag{1}$$

The variables v_i in (1) may be arbitrary state variables in the system model, or linear combinations thereof, and $g(\cdot)$ represents a general function of its arguments. The direct derivation of random-input describing functions (ridf's) for such non-linearities is generally quite tedious ; it is fair to state that the effort involved can be a very real impediment to the application of CADET to systems whose models involve non-linearities of the forms indicated in (1). This note provides a general approach which alleviates this problem to a great extent. Specifically, it is shown that given a quasi-linear representation of the constituent $g(\cdot)$, ridf's for the overall non-linearities indicated in (1) can be obtained with a relatively modest analytic effort.

2. Basic definitions

The ridf's for the non-linearities indicated in (1) are based on the assumption that the input signals v_i are comprised of a deterministic component

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(' mean ') m_i and a random component r_i ; in vector notation,

$$\left.\begin{aligned}\mathbf{m}&=E[\mathbf{v}]\\ \mathbf{r}&=\mathbf{v}-\mathbf{m}\end{aligned}\right\} \tag{2}$$

where $E[\cdot]$ is the expectation operator. The random components are assumed to be jointly normal, with second central moments given by the covariance matrix

$$P=E[\mathbf{r}\mathbf{r}^{\mathrm{T}}] \tag{3}$$

In general, for a non-linear function of several input variables the quasi-linear approximation sought is of the form

$$f(v_1, v_2, \ldots, v_k) \cong \hat{f} + n_1 r_1 + n_2 r_2 + \ldots + n_k r_k \triangleq \hat{f} + \mathbf{n}^{\mathrm{T}}\mathbf{r} \tag{4}$$

where

$$\left.\begin{aligned}\hat{f} &\triangleq \frac{1}{[(2\pi)^k|P|]^{1/2}} \int_{-\infty}^{\infty}\cdots\int f(v_1, v_2, \ldots, v_k) \times \exp\left(-\tfrac{1}{2}\mathbf{r}^{\mathrm{T}}P^{-1}\mathbf{r}\right) dv_1\, dv_2 \ldots dv_k\\ \mathbf{n}^{\mathrm{T}} &\triangleq \frac{1}{[(2\pi)^k|P|]^{1/2}} \int_{-\infty}^{\infty}\cdots\int \mathbf{r}^{\mathrm{T}} f(v_1, v_2, \ldots, v_k) \times \exp\left(-\tfrac{1}{2}\mathbf{r}^{\mathrm{T}}P^{-1}\mathbf{r}\right) dv_1\, dv_2 \ldots dv_k\, P^{-1}\end{aligned}\right\} \tag{5}$$

The definitions (5) retain the exact non-linearity output mean and input–output cross-correlation properties in (4) ; this quasi-linear approximation was first proposed by Booton (1954) in the zero-mean case, and extended to the general case by Somerville and Atherton (1958). The relation above defining the random-component ridf gain vector $\mathbf{n}$ need not be evaluated explicitly, since

$$\mathbf{n}^{\mathrm{T}} = \frac{\partial \hat{f}}{\partial \mathbf{m}} \tag{6}$$

as can be observed by direct differentiation (Phaneuf 1968). Consequently, attention can be primarily focused on the calculation of $\hat{f}$.

3. Main results

The derivation of general ridf's for f_1 in (1) provides a demonstration of the approach for treating all non-linearities that are linear in one input variable. Consider the argument of the exponential factor, $\mathbf{r}^{\mathrm{T}}P^{-1}\mathbf{r}$ in (5) : a linear transformation $\mathbf{w}=R^{-1}\mathbf{r}$ can be used to simplify the integrand. For two variables, choose the matrix R in terms of the elements of P to be

$$P=\begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}, \quad R \triangleq \begin{bmatrix}\sigma_1\sqrt{(1-\rho^2)} & \sigma_1\rho\\ 0 & \sigma_2\end{bmatrix} \tag{7}$$

so that

$$\mathbf{r}^{\mathrm{T}}P^{-1}\mathbf{r} = \mathbf{w}^{\mathrm{T}}\mathbf{w} \tag{8}$$

This change of variables in (5) leads to

$$\hat{f}_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{m_1 + \sigma_1(\sqrt{(1-\rho^2)}w_1 + \rho w_2)\} g(\sigma_2 w_2 + m_2) \times \exp\left[-\tfrac{1}{2}(w_1{}^2 + w_2{}^2)\right] dw_1\, dw_2 \tag{9}$$

The matrix R in (7) is specifically chosen to be lower triangular, i.e. zero below the diagonal, in order to make v_2 a linear function of w_2 alone. This permits integration with respect to w_1 to be carried out in (9) irrespective of the form of g. Integrating with respect to w_1 yields

$$\hat{f}_1 = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} (m_1 + \sigma_1 \rho w_2) g(\sigma_2 w_2 + m_2) \exp\left(-\tfrac{1}{2}w_2{}^2\right) dw_2 \tag{10}$$

which has reduced the evaluation of $\hat{f}_1$ in (9) to an integration in one variable.

The result in (10) can be further interpreted to yield the fundamental form for non-linearities which are linear in one variable. Beginning with the ridf approximation of $g(v_2)$ as defined in (4) and (5), that is,

$$g(v_2) \cong \hat{g} + n_g r_2 \tag{11}$$

it is observed that the transformation used in (7) leads to

$$\left.\begin{aligned} \hat{g} &\triangleq \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} g(\sigma_2 w_2 + m_2) \exp\left(-\tfrac{1}{2}w_2{}^2\right) dw_2 \\ n_g &\triangleq \frac{1}{\sigma_2{}^2} \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \sigma_2 w_2\, g(\sigma_2 w_2 + m_2) \exp\left(-\tfrac{1}{2}w_2{}^2\right) dw_2 \end{aligned}\right\} \tag{12}$$

Comparing (10) and (12), it is recognized that

$$\left.\begin{aligned} \hat{f}_1 &= m_1 \hat{g} + \rho \sigma_1 \sigma_2 n_g \\ &= m_1 \hat{g} + p_{12} n_g \end{aligned}\right\} \tag{13}$$

The two random component ridf's in (4) are found from (6) to be

$$\left.\begin{aligned} n_1 &= \frac{\partial \hat{f}_1}{\partial m_1} = \hat{g}(m_2, \sigma_2) \\ n_2 &= \frac{\partial \hat{f}_1}{\partial m_2} = m_1 n_g + p_{12} \frac{\partial n_g}{\partial m_2} \end{aligned}\right\} \tag{14}$$

Consequently, given that the non-linearity $g(v_2)$ is readily quasi-linearized (and ridf's for a large variety of one-input non-linearities are available in Gelb and Vander Velde (1968)), it is a direct matter of differentiation to evaluate ridf's for the multi-input non-linearity $v_1 g(v_2)$.

By using similar transformation techniques (refer to (7)), the three-variable case

$$f_2(v_1, v_2, v_3) = v_1 g(v_2, v_3) \tag{15}$$

has been proven to lead to a mean component quasi-linear term of the form

$$\left.\begin{aligned}\hat{f}_2 &= m_1\hat{g} + p_{12}\frac{\partial\hat{g}}{\partial m_2} + p_{13}\frac{\partial\hat{g}}{\partial m_3}\\ &= m_1\hat{g} + p_{12}n_{g_2} + p_{13}n_{g_3}\end{aligned}\right\} \tag{16}$$

where the ridf approximation of $g(v_2, v_3)$ is

$$g(v_2, v_3) \cong \hat{g} + n_{g_2}(v_2 - m_2) + n_{g_3}(v_3 - m_3) \tag{17}$$

This result should greatly expedite the evaluation of ridf's for three-input non-linearities that are linear in one variable.

The results described above by (13) and (16) can be directly extended to derive a general direct quasi-linear approximation for the non-linearity class

$$f_3(v_1, v_2) = v_1{}^k g(v_2), \quad k = 2, 3, \ldots \tag{18}$$

First, consider

$$f(v_1, v_2) = v_1{}^2 g_2(v_2) \tag{19}$$

as a special case of (15) and (16) with g given by $v_1 g_2(v_2)$. Applying (13) yields

$$\hat{f} = (m_1{}^2 + \sigma_1{}^2)\hat{g}_2 + 2m_1 p_{12}\frac{\partial\hat{g}_2}{\partial m_2} + p_{12}{}^2\frac{\partial^2\hat{g}_2}{\partial m_2{}^2} \tag{20}$$

One can then proceed by induction to show that the general form of the mean component ridf for f_3 in (1) is

$$\hat{f}_3 = \sum_{j=0}^{k}\binom{k}{j} p_{12}{}^j E[v_1{}^{k-j}]\frac{\partial^j\hat{g}}{\partial m_2{}^j} \tag{21}$$

where $\binom{k}{j}$ is the standard binomial coefficient notation

$$\binom{k}{j} \triangleq \frac{k(k-1)\ldots(k-j+1)}{j!} \tag{22}$$

The random component ridf's are directly obtained by differentiation according to (6). This simple and powerful expression for $\hat{f}_3$ reduces the ridf evaluation to a relatively easy task for a broad class of two-input non-linearities.

4. Examples

Example 1

For the non-linearity

$$f(v_1, v_2) = v_1 \sin v_2 \tag{23}$$

the result

$$E[\sin v_2] = \exp(-\tfrac{1}{2}p_{22}) \sin m_2 \tag{24}$$

(Gelb and Vander Velde 1968) and (13) and (14) lead to

$$\left.\begin{aligned}\hat{f} &= [m_1 \sin m_2 + p_{12}\cos m_2]\exp(-\tfrac{1}{2}p_{22})\\ n_1 &= \exp(-\tfrac{1}{2}p_{22})\sin m_2\\ n_2 &= [m_1\cos m_2 - p_{12}\sin m_2]\exp(-\tfrac{1}{2}p_{22})\end{aligned}\right\} \tag{25}$$

Example 2

For a non-linear function with multiple trigonometric factors, e.g.

$$f(v_1, v_2, v_3) = v_1 \sin v_2 \cos v_3 \tag{26}$$

the standard sum-and-difference formulae lead to

$$g \triangleq \sin v_2 \cos v_3 = \tfrac{1}{2}[\sin(v_2+v_3) + \sin(v_2-v_3)]$$

Application of the result (24) yields

$$\left.\begin{aligned} \hat{g} &= E[\sin v_2 \cos v_3] \\ &= \tfrac{1}{2} \exp[-\tfrac{1}{2}(p_{22}+p_{33})][\exp(-p_{23}) \sin(m_2+m_3) \\ &\qquad + \exp(p_{23}) \sin(m_2-m_3)] \end{aligned}\right\} \tag{27}$$

and the direct use of (16) leads to

$$\hat{f} = m_1\hat{g} + \tfrac{1}{2} \exp[-\tfrac{1}{2}(p_{22}+p_{33})][(p_{12}+p_{13}) \exp(-p_{23}) \cos(m_2+m_3) + (p_{12}-p_{13}) \exp(p_{23}) \cos(m_2-m_3)] \tag{28}$$

Obtaining this result by the direct application of (5) would be quite time-consuming.

5. Conclusion

With the tools developed in this note and a catalogue of single-variable ridf's such as that provided in Gelb and Vander Velde (1968), a broad class of non-linearities can be treated in a straightforward manner (with little or no analysis of the sort illustrated in (5) to (13)). In particular, these results obviate performing integrations over several variables as called for in (5), which is generally an onerous task. It is the author's opinion that similar results can be obtained for analogous, more complicated multi-input non-linearities (e.g. $v_1{}^k g(v_2, v_3)$, $v_1 g(v_2, v_3, v_4)$, etc.). These contributions significantly enhance the usefulness of CADET, and any other analytic techniques based on ridf theory.

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