Statistical Performance Analysis of Nonlinear Stochastic Systems

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Abstract. This paper presents and contrasts two methods that can be used to generate approximate time histories of the second-order statistics of the state variables of nonlinear stochastic systems: the monte carlo method [1] and covariance analysis based on statistical linearization [2]. In both cases, it will be shown that these statistics can be determined quite accurately if the variables are reasonably gaussian, where the measure of the gaussianness of a random variable is its kurtosis, or the ratio of the fourth central moment to the square of the variance. The first technique has the advantage of better error diagnostic capability, while the second generally is much less expensive to use, in terms of computer time consumption. If the variables of a stochastic system are quite nongaussian, then second-order statistics are most generally not meaningful; the monte carlo method can then be used to generate approximate cumulative distributions. The utilization and efficacy of the two methods are illustrated in a number of applications.

1. Introduction and Problem Statement. The dynamics of many nonlinear continuous-time stochastic systems can be written in the form of a state vector differential equation,

$$\dot{\underline{z}} = \underline{f}(\underline{z},\underline{y},t) \tag{1}$$

where $\underline{z}$ is the vector of system states, $\underline{y}$ is a vector of random inputs, and $\underline{f}(\underline{z},\underline{y},t)$ represents the nonlinear time-varying dynamic relationships in the system. Often the elements of $\underline{y}$ are correlated random processes with deterministic components that may be nonzero, so that a system model of the form

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$$\dot{\underline{x}} = \underline{f}(\underline{x},t) + G(t)\ \underline{w}(t) \tag{2}$$

may be used, where $\underline{x}$ is an augmented state vector, $\underline{x}^T = [\underline{z}\,|\,\underline{y}]^T$, and $\underline{w}$ is the sum of a vector of white noise processes and a deterministic vector which serves as the input to generate the random vector $\underline{y}$. Henceforth, we treat (2) as the basic system model.

The initial condition of the state vector is specified by assuming that the state variables are jointly normal. Thus, the initial condition is completely specified by an initial mean vector and covariance matrix,

$$E\ [\underline{x}(0)1] = \underline{m}_0,\ E\ [(x(0) - m_0)(x(0) - m_0)^T] = S_0 \tag{3}$$

As stated above, the input vector $\underline{w}$ is assumed to be composed of elements that are gaussian white noise processes, plus an additive deterministic component or mean; thus

$$E\ [\underline{w}(t)] = \underline{b}(t),\ E\ [\underline{w}(t)-\underline{b}(t))\ (\underline{w}(\tau)-\underline{b}(\tau))^T] = Q(t)\ \delta(t-\tau) \tag{4}$$

where $Q(t)$ is the input spectral density matrix. Given the initial condition and input statistics defined above, and the system model formulated in (2), our usual goal is to determine the state variable second-order statistics $\underline{m}(t)$, $S(t)$ over a time period of interest.

2. The Monte Carlo Method. This technique provides a straightforward approach to the statistical analysis of the performance of a nonlinear system with random inputs, based on direct simulation. It entails determining the system response to a finite number of "typical" initial conditions and random input functions which are generated according to their specified statistics. Given the information outlined in Section 1, monte carlo analysis requires a large number, say q, of representative simulations of the system response. Performing q independent simulations yields an ensemble of state trajectories, denoted $\underline{x}^{(i)}$, i=1, 2, ..., q. The mean $\underline{m}(t)$ and covariance $S(t)$ of the state vector are estimated by averaging over the ensemble of trajectories using the relations

$$\hat{\underline{m}}(t) \triangleq \frac{1}{q} \sum_{i=1}^{q} \underline{x}^{(i)}(t) \cong \underline{m}(t),$$

(5)

$$\hat{S}(t) \triangleq \frac{1}{q-1} \sum_{i=1}^{q} (\underline{x}^{(i)} - \hat{\underline{m}})(\underline{x}^{(i)} - \hat{\underline{m}})^T \cong S(t)$$

where $\hat{\underline{m}}(t)$ and $\hat{S}(t)$ denote the resulting sample statistics.

In order to assess the accuracy of the approximate statistics given in (5), it is necessary to consider the statistical properties of the estimates $\hat{\underline{m}}(t)$ and $\hat{S}(t)$. To simplify the notation, consider a scalar random variable u (e.g., the value of some system state variable at some time of interest), and let m and σ represent the true values of the mean and standard deviation of u,

$$m = E[u], \quad \sigma^2 = E[(u - m)^2] \tag{6}$$

By performing one set of q monte carlo trials, we obtain a single estimate of m and σ, denoted $\hat{m}$ and $\hat{\sigma}$. These estimates are also random variables; that is, if another set of q monte carlo trials were performed independently of the first set, then different estimates for the mean and variance would be obtained. Thus, the accuracy of the sample statistics must be characterized statistically.

The usual approach is to determine <u>confidence limits</u>, $\underline{m}$, $\overline{m}$, $\underline{\sigma}$ and $\overline{\sigma}$ so that

$$\text{Prob}\,[\underline{m} \le m \le \overline{m}] = \psi, \quad \text{Prob}\,[\underline{\sigma} \le \sigma \le \overline{\sigma}] = \psi \tag{7}$$

where ψ is generally nearly unity (0.95 or 0.99, for example). The confidence limits for m are obtained directly from the sample statistics [1]:

$$\underline{m} = \hat{m} - n_\sigma \frac{\hat{\sigma}}{\sqrt{q}}, \quad \overline{m} = \hat{m} + n_\sigma \frac{\hat{\sigma}}{\sqrt{q}} \tag{8}$$

where n_σ is chosen to correspond to the desired degree of confidence; since $\hat{m}$ is nearly gaussian for q reasonably large*, n_σ satisfies:

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*The best rule-of-thumb for determining what q is reasonably large is that $q > 25(\lambda - 1)$; refer to next page.

$$(9)\qquad \frac{1}{\sqrt{2\pi}}\int_{-n_\sigma}^{n_\sigma} \exp\left(-\tfrac{1}{2}\zeta^2\right)d\zeta = \psi \longrightarrow \boxed{\begin{array}{l}\psi = 0.90, n_\sigma = 1.645\\ \psi = 0.95, n_\sigma = 1.960\\ \psi = 0.99, n_\sigma = 2.576\end{array}}$$

The confidence limits for σ are not completely determined by the sample statistics unless the kurtosis, λ, is known:

$$\lambda \triangleq \mu_4/\sigma^4,\quad \mu_4 = E\left[(u-m)^4\right]$$

Then, for q reasonably large* [1]:

$$(10)\qquad \underline{\sigma} = \frac{\hat{\sigma}}{\sqrt{1 + n_\sigma\sqrt{\frac{\lambda-1}{q}}}} \triangleq \underline{\rho}\,\hat{\sigma},\quad \overline{\sigma} = \frac{\hat{\sigma}}{\sqrt{1 - n_\sigma\sqrt{\frac{\lambda-1}{q}}}} \triangleq \overline{\rho}\,\hat{\sigma}$$

Typical values of λ for various density functions of u are: gaussian, $\lambda = 3$; exponential, $\lambda = 6$; triangular, $\lambda = 2.4$; uniform, $\lambda = 1.8$; and bipolar, $\lambda = 1.0$. Observe that the confidence limits for σ are expressed as dimensionless multipliers, $\underline{\rho}$ and $\overline{\rho}$, times the sample standard deviation. Example: For 95% confidence limits of about $\pm$ 10% ($\underline{\rho} = 0.92$, $\overline{\rho} = 1.10$) 256 trials are required for gaussian random variables ($\lambda = 3$). A more complete discussion of the variation of the limit multipliers with varying q, ψ, and λ with accompanying tables, may be found in [1].

3. Covariance Analysis Using Statistical Linearization. The differential equations that govern the propagation of the mean vector and covariance matrix for the system described by (2) can be derived directly [3]:

$$(11)\qquad \begin{aligned}\dot{m} &= E\left[\underline{f}(\underline{x},t)\right] + G(t)\underline{b} \triangleq \hat{\underline{f}} + G(t)\underline{b}\\ \dot{S} &= E\left[\underline{f}\,\underline{r}^T\right] + E\left[\underline{r}\,\underline{f}^T\right] + G(t)QG^T(t)\end{aligned}$$

The equation for S can be put into a form analogous to the covariance equations corresponding to $\underline{f}$ being linear, by defining the auxiliary matrix N through the relationship

$$(12)\qquad NS \triangleq E\left[\underline{f}(\underline{x},t)\,\underline{r}^T\right]$$

*See footnote on the preceding page.

Then (11) may be written as

$$\dot{\underline{m}} = \hat{\underline{f}} + G(t)\underline{b}, \quad \dot{S} = NS + SN^T + G(t)QG^T(t) \tag{13}$$

The quantities $\hat{\underline{f}}$ and N defined in (11) and (12) must be determined before (13) can be solved. Evaluating the indicated expected values exactly requires knowledge of the joint probability density function (joint pdf) of the state variables. The fact that the pdf is generally not available thus precludes the exact solution of (13).

The procedure for obtaining an approximate solution to (13) is to assume the form of the joint probability density function of the state variables in order to evaluate $\hat{\underline{f}}$ and N according to (11) and (12). Although it is possible to use any joint pdf, it is both reasonable and convenient to assume that the state variables are jointly normal.

As a consequence of the gaussian assumption, which is discussed more fully in [2] and [4], the elements of $\hat{\underline{f}}$ and N are random-input describing functions [5,6], which for a given nonlinearity are dependent only upon the mean and the covariance of the system state vector. Furthermore, it can be proved [7] that

$$N(\underline{m},S,t) = \frac{d}{d\underline{m}} \hat{\underline{f}}(\underline{m},S,t) \tag{14}$$

Since calculating $\hat{\underline{f}}$ is required for the propagation of the mean, it is generally much easier to employ (14) than to evaluate N directly using (12). The random-input describing functions (RIDF's) for a wide variety of nonlinearities are immediately available from [5, 6, 8], so generally little effort is required in obtaining $\hat{\underline{f}}$ and N.

The combination of covariance analysis with describing function theory to provide approximate solutions to (13) was originated in [9, 10]. For succinctness, this approach is called CADET™ -- the Covariance Analysis Describing Function Technique [10].

A comparison of statistical linearization with the classical Taylor series or small-signal linearization technique, wherein $\underline{f}(\underline{m})$ and

*The term CADET is a trademark of The Analytic Sciences Corporation.

the Jacobian $J = \underline{df}/\underline{dm}$ are used instead of $\hat{\underline{f}}$ and N, provides a great deal of insight into the success of RIDF's in capturing nonlinear effects. If a saturation or limiter is present in a system and its input x is zero-mean, for example, the Taylor series approach leads to replacing f(x) with a unity gain regardless of the input amplitude, while the corresponding RIDF [5,6] decreases as the rms value of x, σ_x, increases, thus accurately reflecting the nonlinear effect that is neglected in the small-signal linear model. The fact that RIDF's retain an essential characteristic of system nonlinearities -- input-amplitude dependence -- provides the basis for the proven accuracy of statistical linearization.

4. Illustrative Examples. The following results demonstrate the two techniques of statistical performance analysis, and illustrate many of the accompanying comments about their use and accuracy.

4.1 A missile/target intercept problem. A variable of particular interest in the planar missile-target intercept problem (cf. [11]) during the terminal homing phase is the cross-range (lateral) separation between the missile and target, denoted y. In a typical analysis, y (and all other system variables) is assumed to be gaussian at the initiation of the terminal homing phase, and y remains quite gaussian until the last few seconds of the engagement. Figure 1 shows the variation of σ_y with time during a six-second engagement ensemble, where a quite highly nonlinear system model [11] has been used for simulation purposes. The solid curve is obtained by CADET, and the results of a 500-trial monte carlo study are indicated with circled data points to indicate $\hat{\sigma}_y$ and vertical I-bars to indicate the 95% confidence limits. The estimated value of kurtosis is also indicated near each data point; as observed above, $\hat{\lambda}$ is nearly 3 until the last second, while at the final time, t=6 sec, $\hat{\lambda} \cong 15$, which is indicative of the quite highly gaussian character of the final lateral separation ("miss distance").

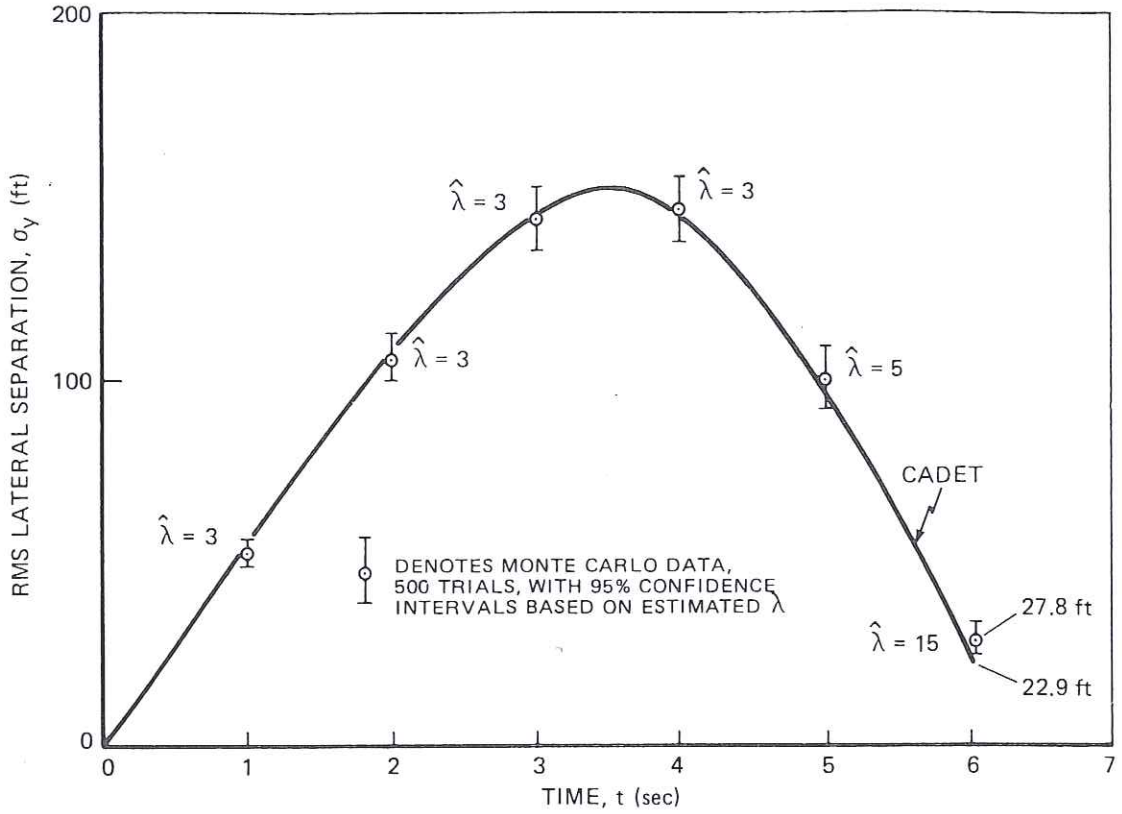
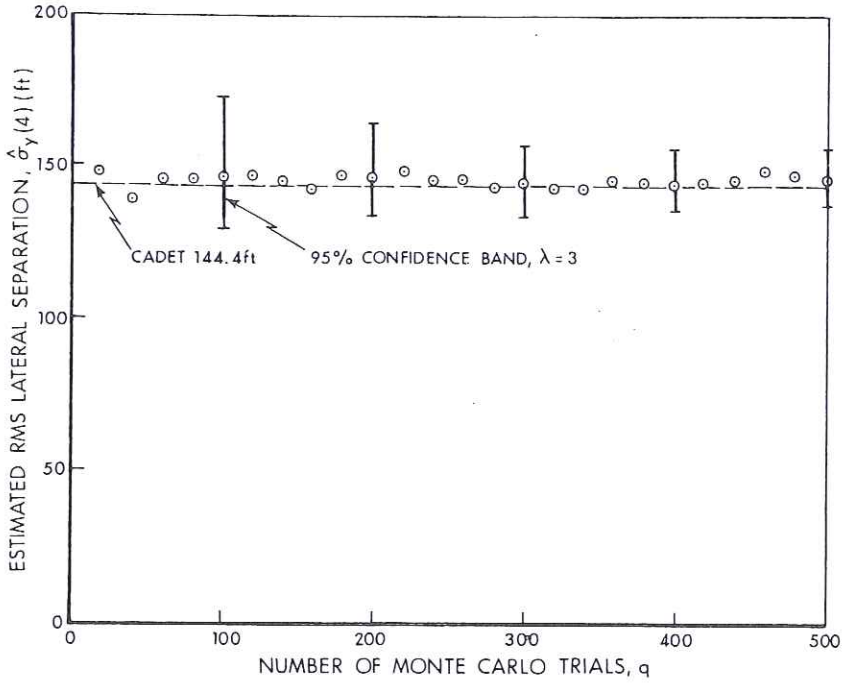


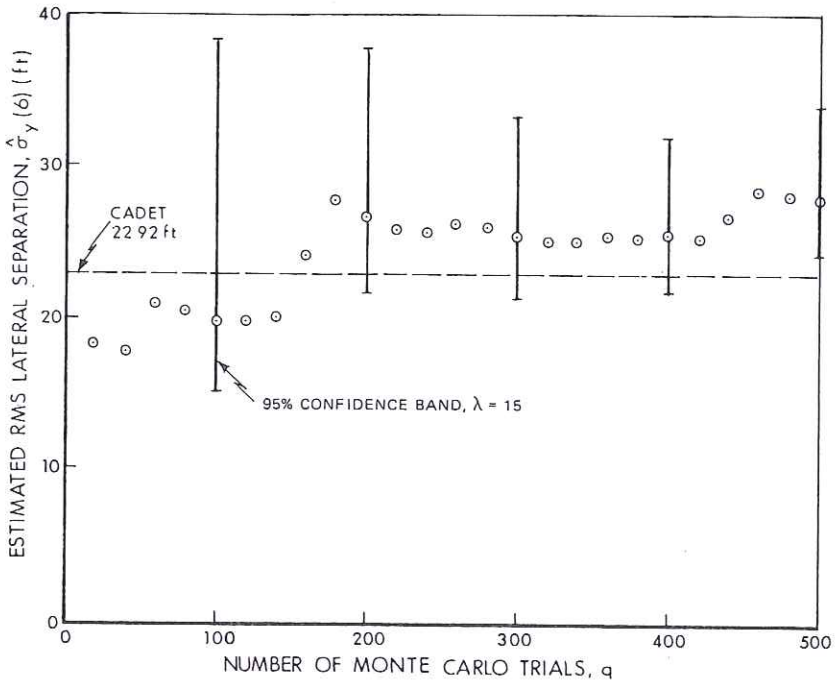
Figure 1. Time History of rms Missile-Target Lateral Separation.

Figure 2 gives a more detailed view of the CADET and monte carlo analyses depicted in Figure 1; for two values of time the estimated σ_y is shown as a function of the number of trials performed, q. We note in Figure 2 that the estimated value of σ_y at t=4 appears to "settle" to about 145 ft after a few hundred trials, while $\hat{\sigma}_y(6)$ converges much more slowly, as (10) predicts. The CADET result appears to be excellent at t=4 sec, and is certainly as accurate as several hundred monte carlo trials at t=6 where y is quite highly nongaussian.

4.2 An antenna pointing and tracking problem. The purpose of the antenna pointing and tracking system modeled in Fig. 3 is to follow a target line-of-sight (LOS) angle, θ_t. Assume that θ_t is a deterministic ramp, with an angular rate of ω degrees per second. The pointing error, $e = \theta_t - \theta_a$, where θ_a is the antenna centerline angle, is the input to a nonlinearity $f(\cdot)$ which represents the limited beamwidth of the antenna; for the present discussion,



(a) Time t = 4 sec.



(b) Time t = 6 sec.

Figure 2. Comparison of CADET and Monte Carlo rms Lateral Separation

$$f(e) = e(1-k_a e^2)$$

where k_a is suitably chosen to represent the antenna characteristic.

The noise w(t) injected at the receiver is a white noise process having zero mean and spectral density q. This problem formulation is taken directly from [12]; a more thorough discussion of the approach

and results in [12] vis-à-vis the current treatment is given in [13]. In the form given in (2), the system is governed by $\dot{\underline{x}} = \underline{f}(\underline{x}) + \underline{w}$, where

$$\underline{f}(\underline{x}) = \begin{bmatrix} -kx_2 \\ a[f(x_1) - x_2] \end{bmatrix}, \qquad \underline{w} = \begin{bmatrix} \dot{\theta}_t \\ aw(t) \end{bmatrix}$$

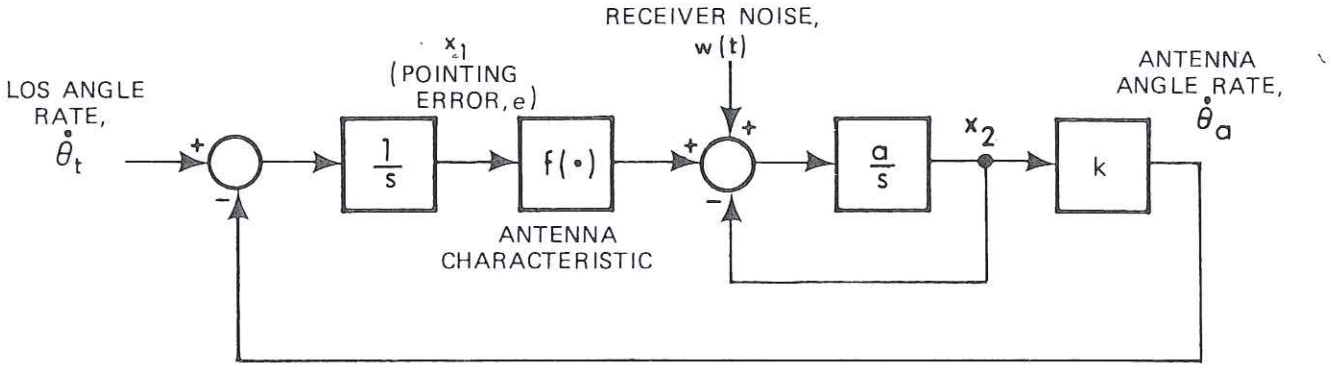


Figure 3. Antenna Pointing and Tracking Model

The solutions depicted in Fig. 4 are based on the assumption that e(0) and w(t) are gaussian. The system parameters are: $a = 50\ \text{sec}^{-1}$, $k = 10\ \text{sec}^{-1}$, $k_a = 0.4\ \text{deg.}^{-2}$, $m_{e0} = 0.4$ deg, $\sigma_{e0} = 0.1$ deg, $q = 0.004\ \text{deg}^2$. The goal is to determine tracking capability for various values of ω; for brevity, only $\omega = 6$ deg/sec is shown here. Four solutions are presented: CADET results; ensemble statistics from a 200-trial monte carlo simulation, with 95% confidence intervals based on the gaussian assumption; results using the linear approximation (corresponding to $k_a = 0$); and the second-order Volterra series solution based on a technique given in [12]. The CADET solution provides a significantly better agreement with the monte carlo data than either of the other analytical approximations, which failed to capture the instability that occurs when tracking error is large, thus clearly establishing the superiority of statistical linearization.

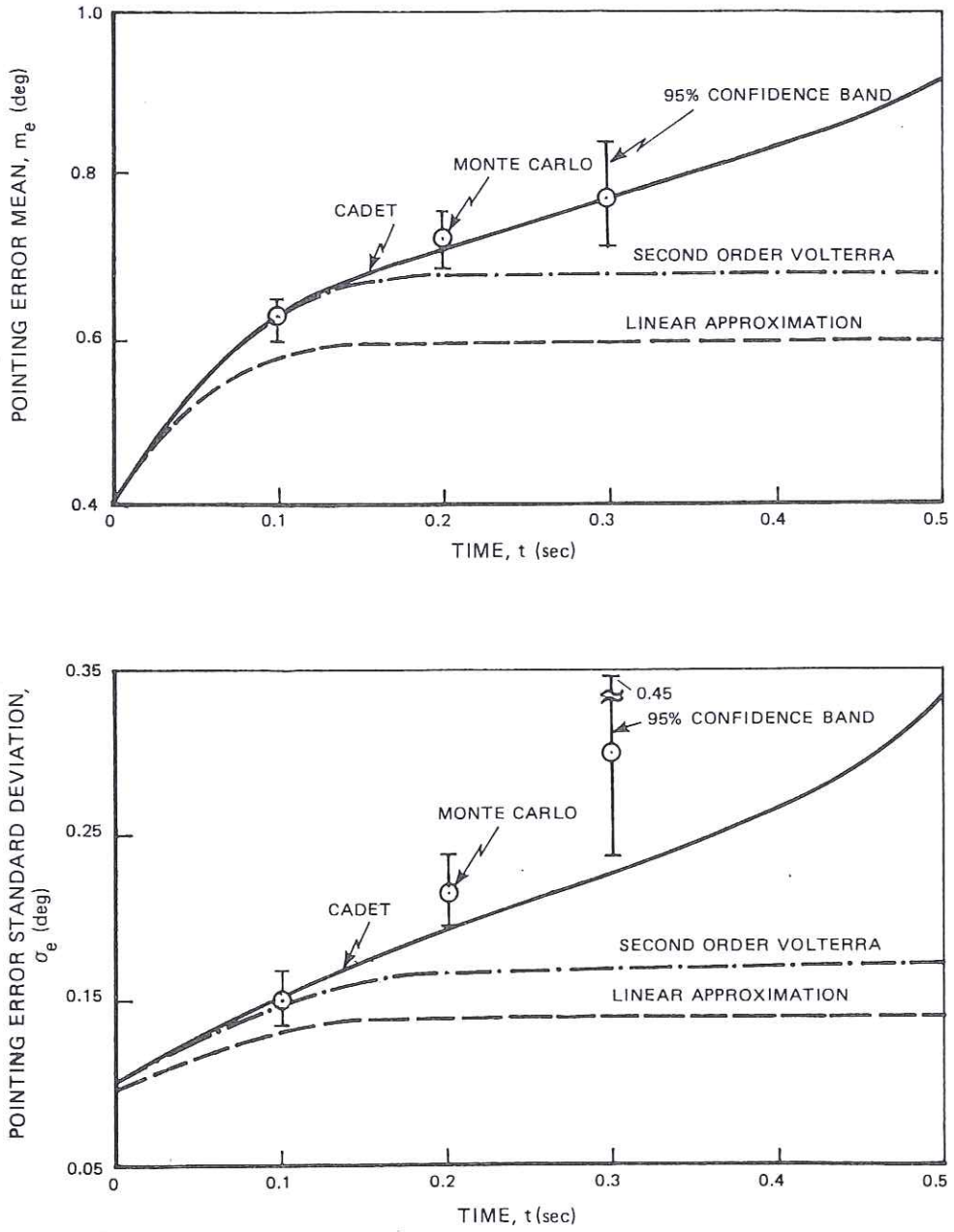


Figure 4. Pointing error statistics for $\omega = 6$ deg/s.

5. Conclusion. The two best methods now available for the statistical performance analysis of nonlinear systems have been outlined. Both are approximate, although the monte carlo method can be made arbitrarily accurate by performing enough trials. Since it is necessary to estimate kurtosis quite well in order to assess this accuracy, the trials required may be excessive [1]. If kurtosis is reasonable (experience suggests that λ should be less than 10 to 15), then CADET produces results that are comparable to 250- to 500-trial monte carlo studies with a small fraction of the computer time expenditure. (Savings of 90 to 98% have been achieved.)

For larger values of λ, the significance of second-order statistics is largely lost, unless one knows the pdf of the variable; one's only recourse is to use the monte carlo method to generate histograms to approximate the cumulative distribution of the random variable. The meaninglessness of $\hat{\sigma}$ as a performance measure is illustrated in Figure 5, which shows an empirical miss distance distribution plot with standard deviation σ = 1018 ft and kurtosis of about 25; clearly the missile under consideration performs much better than this statistic suggests, as the "gaussian fit" curve for this value of rms miss distance shows.

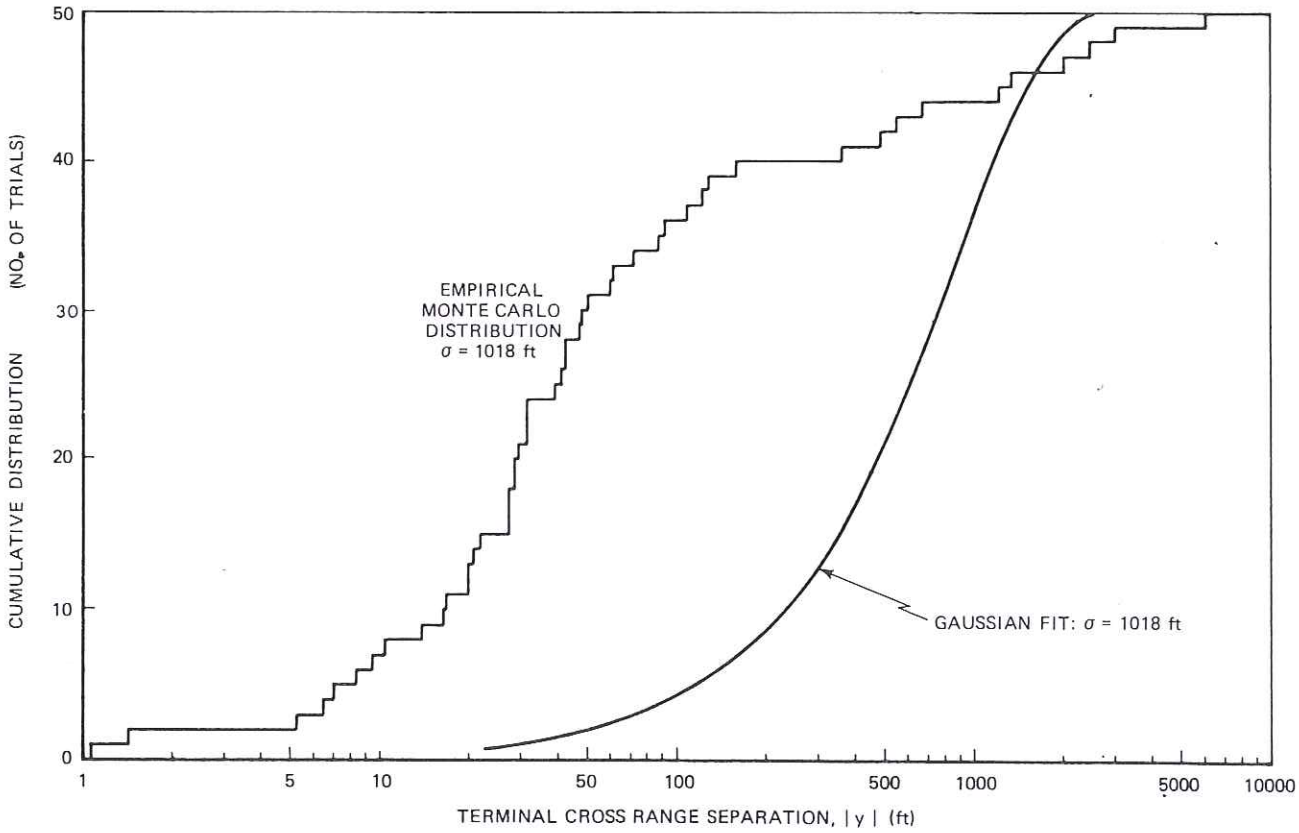


Figure 5. Empirical Probability Distribution Plot for Tactical Missile Miss Distance.

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