A General Limit Cycle Analysis Method for Multivariable Systems

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Abstract. The sinusoidal-input describing function (SIDF) technique is a well-known approach for studying limit cycle phenomena in non-linear systems with one nonlinearity [1,2]. In recent years, a number of extensions of the SIDF method have been developed to permit the analysis of systems containing more than one nonlinearity. In most cases, the nonlinear system models that can be treated by such extensions have been quite restrictive (limited to a few nonlinearities, or to certain specific configurations). Furthermore, some results involve only conservative conditions for limit cycle avoidance, rather than actual limit cycle conditions. The technique described in this paper removes all constraints: Systems described by a general state vector differential equation, with any number of nonlinearities, may be analyzed. In addition, the nonlinearities may be multi-input, and bias effects can be treated.

The general SIDF approach was first fully developed in [3], and its power and use were illustrated by application to a highly nonlinear model of a tactical aircraft in a medium angle-of-attack flight regime [4,5]. Some problems associated with direct simulation (especially "obscuring modes" and the initial condition problem) were also discussed in [5]. This presentation highlights the basic results from [5], and treats a new application (bifurcations in a two-mode panel flutter model) in detail.

1. Introduction. The study of limit cycle (LC) conditions in non-linear systems is a problem of considerable interest in engineering. An approach to LC analysis that has gained widespread acceptance is the frequency domain/sinusoidal-input describing function (SIDF) method [1,2]. This technique, as it was first developed for systems with one dominant nonlinearity, involved formulating the system in

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the following form:

$$(1)\quad \begin{aligned} \dot{\underset{\sim}{x}} &= F\underline{x} + \underline{g}\upsilon \\ \upsilon &= -\phi(\sigma) \\ \sigma &= \underline{h}^T\underline{x} + \kappa\upsilon \end{aligned}$$

where $\underset{\sim}{x}$ is an n-dimensional state vector. There is thus one single-input/single-output (SISO) nonlinearity, $\phi(\sigma)$, and linear dynamics of arbitrary order that may be represented by the SISO transfer function (in Laplace transform notation) $W(s) = \underline{h}^T(sI-F)^{-1}\,\underline{g} + \kappa$.

It is then assumed that the input σ may be essentially sinusoidal, e.g., $\sigma = a \cos \omega t$, and the output approximation

$$(2)\quad \begin{aligned} \phi(\sigma) &\cong \text{Re}\,[\psi_1 \exp\,(i\omega t)] \\ &\triangleq \text{Re}\,[n_1(a) * a \exp\,(i\omega t)] \end{aligned}$$

is made*. The fourier coefficient† ψ_1 (and thus the "gain" n_1) is generally complex unless $\phi(\sigma)$ is single valued; the real and imaginary parts of ψ_1 represent the in-phase (cosine) and quadrature (-sine) fundamental components of $\phi(a \cos \omega t)$, respectively. The so-called describing function $n_1(a)$ in (2) is "amplitude dependent", thus retaining a basic property of a nonlinear operation. By the principle of harmonic balance, the assumed oscillation -- if it is to exist -- must result in a linearized system with pure imaginary eigenvalues,

$$|i\omega - F + n_1\, \underline{g}\underline{h}^T| = 0$$

for some value of ω, or by elementary matrix operations

$$(3)\quad W(i\omega) = -\,1/n_1(a)$$

Condition (3) is easy to verify using the polar or Nyquist plot of $W(i\omega)$ [1,2]; in addition, the LC amplitude a is determined in the process.

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*If $\phi(\sigma)$ is not odd ($\phi(-\sigma) \neq -\phi(\sigma)$), a constant term ("bias" or "D.C. value") must occur in (2); such cases present no difficulty [1,2], but are omitted to simplify the discussion.

†The usual definition of an SIDF is that $n_1(a)$ is chosen to minimize the mean square error between $f(a \cos \omega t)$ and $\text{Re}\,[n_1(a) * a \exp\,(i\omega t)]$; thus a $n_1(a)$ is the first fourier coefficient [1,2].

It is generally well-understood that SIDF analysis as outlined above is only approximate, so caution is always recommended in its use. The standard caveats that $W(i\omega)$ should be "low pass to attenuate higher harmonics" and that $\phi(\sigma)$ should be "well-behaved" (so that the first harmonic in (2) is dominant) indicate that the analyst has to be familiar with the system behavior, by direct experience or by simulation. Given an appreciation of these warnings, SIDF LC analysis has proven to be a very powerful engineering tool.

The utility of SIDF analysis for systems with one significant SISO nonlinearity as outlined above has naturally resulted in a number of attempts to generalize the technique to the multiple-nonlinearity case. In most cases that preceded [3], only SISO nonlinearities were considered, and bias effects (either due to constant inputs or to "rectification" caused by nonlinear effects) were excluded. Also, special model configurations were often assumed. The earlier results are discussed more fully in [5]. The LC analysis approach described in this paper removes all restrictions with respect to model configuration, nonlinearity type, or the presence of biases.

2. The General SIDF Limit Cycle Analysis Method. The most general system model considered here is

$$(4) \qquad \dot{\underline{x}} = \underline{f}\ (\underline{x},\underline{u})$$

when $\underline{x}$ is an n-dimensional state vector and $\underline{u}$ is an n-dimensional input vector. Assuming that $\underline{u}$ is a vector of constants, denoted $\underline{u}_o$, it is desired to determine if (4) may exhibit LC behavior.

As before, we assume that the state variables are nearly sinusoidal,

$$(5) \qquad \underline{x} \cong \underline{x}_c + \text{Re}\ (\underline{a}\ \exp\ (i\omega t)]$$

where $\underline{a}$ is a complex amplitude vector and $\underline{x}_c$ is the state vector center value (which is not a singularity, or solution to $\underline{f}\ (\underline{x}_o,\underline{u}_o) = \underline{0}$ unless the nonlinearities satisfy certain stringent symmetry conditions with respect to $\underline{x}_o$). Then we again assume that higher harmonics are negligible, to make the approximation

(6) $\underline{f}(\underline{x},\underline{u}_o) \cong \underline{f}_{DF}(\underline{u}_o,\underline{x}_c,\underline{a}) + \mathrm{Re}\,[F_{DF}(\underline{u}_o,\underline{x}_c,\underline{a})\ \underline{a}\ \exp(i\omega t)]$

The real vector $\underline{f}_{DF}$ and the (generally complex) matrix F_{DF} are obtained by taking the fourier expansions of the elements of $\underline{f}(\underline{x}_c + \mathrm{Re}\ \underline{a}\ \exp(i\omega t), \underline{u}_o)$, and provide the quasi-linear or describing function representation of the nonlinear dynamic relation. The assumed limit cycle exists if $\underline{x}_c$ and $\underline{a}$ can be found so that

(7) (i) $\underline{f}_{DF}(\underline{u}_o,\underline{x}_c,\underline{a}) = \underline{0}$

(ii) $[i\omega I - F_{DF}(\underline{u}_o,\underline{x}_c,\underline{a})]\underline{a} = 0,\ \underline{a} \neq \underline{0}$

(F_{DF} has a pair of pure imaginary eigenvalues, and $\underline{a}$ is the corresponding eigenvector.)

The nonlinear algebraic equations (7) are generally difficult to solve. An iterative method, based on successive approximation, has been used successfully for a ninth-order, highly nonlinear DE [5].

3. Illustrations.

Example 1. The general SIDF representation of a multi-input nonlinearity is illustrated as follows:

$$\begin{aligned} f_5(x) = x_1x_2^3 &\cong [x_{c1}\,x_{c2}^3 + \tfrac{3}{2}\,x_{c2}\,(x_{c1}\,r_{22} + x_{c2}\,r_{12}) + \tfrac{3}{8}\,r_{12}\,r_{22}] \\ &+ [x_{c2}^3 + \tfrac{3}{4}\,x_{c2}\,r_{22}]\ \mathrm{Re}\,[a_1\exp(i\omega t)] \\ &+ [3x_{c1}\,x_{c2}^2 + \tfrac{3}{4}\,x_{c1}\,r_{22} + \tfrac{3}{2}\,x_{c2}\,r_{12}]\ \mathrm{Re}\,[a_2\exp i\omega t)] \\ &\triangleq f_{5DF} + f_{5,1}\ \mathrm{Re}\,[a_1\exp(i\omega t)] + f_{5,2}\ \mathrm{Re}\,[a_2\exp i\omega t)] \end{aligned}$$

where, denoting the conjugate of a_j by a_j^*,

$r_{ij} = \mathrm{Re}\,[a_i a_j^*] \quad i,j = 1,2$

The above result is obtained by substituting for $\underline{x}$ using (5), applying trigonometric identities and discarding the higher harmonic forms. The quantity f_{5DF} is the (hypothetical) fifth element of $\underline{f}_{DF}$, and $f_{5,1}$, $f_{5,2}$ become entries of F_{DF}. By contrast, if Taylor series or "small-signal" linearization is used, the approximation is

$$\begin{aligned} f_5(x) = x_1x_2^3 &\cong x_{o1}\,x_{o2}^3 + x_{o2}^3\ \mathrm{Re}\,[a_1\exp(i\omega t)] \\ &+ 3x_{o1}\,x_{o2}^2\ \mathrm{Re}\,[a_2\exp(i\omega t)] \end{aligned}$$

While this representation is much simpler, it is only realistic when a_1 and a_2 are small.

Example 2. The following second-order differential equation has been derived to describe the local behavior of solutions to a two-mode panel flutter model [6,7]:

$$\ddot{x} + (\alpha + x^2)\,\dot{x} + (\beta + x^2)\,x = 0 \tag{8}$$

Heuristically, it appears that limit cycles may occur for α negative (so that damping is negative for small values of x but positive for large values). Observe also that there are three singularities if β is negative: $x_o = 0, \pm\sqrt{-\beta}$. The corresponding state vector DE is

$$\dot{\underline{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} \underline{x} - \begin{bmatrix} 0 \\ x_1^2(x_1 + x_2) \end{bmatrix} \tag{9}$$

The SIDF assumption is that

$$x_1 = x = x_c + a_1 \cos \omega t$$

$$x_2 = \dot{x} = -a_1\omega \sin \omega t$$

(From the relation $x_2 = \dot{x}_1$, it is clear that x_2 has no center value, and that $a_2 = i\omega a_1$ in (5)). Therefore, the combined nonlinearity in (9) is quasi-linearized to be

$$\begin{aligned} x_1^2(x_1 + x_2) &= (x_c + a_1 \cos \omega t)^2\,(x_c + a_1 \cos \omega t - a_1\,\omega \sin\omega t) \\ &\cong (x_c^3 + \tfrac{3}{2}\,a_1^2\,x_c) + (3x_c^2 + \tfrac{3}{4}\,a_1^2)\,a_1 \cos \omega t \\ &\quad + (x_c^2 + \tfrac{1}{4}\,a_1^2)(-a_1\,\omega \sin \omega t) \end{aligned}$$

Therefore, the conditions of (7) require that

$$\underline{f}_{DF} = \begin{bmatrix} 0 \\ -x_c\,(\beta + x_c^2 + \tfrac{3}{2}\,a_1^2) \end{bmatrix} = \underline{0} \tag{10}$$

$$F_{DF} = \begin{bmatrix} 0 & 1 \\ -(\beta + 3x_c^2 + \tfrac{3}{4}\,a_1^2) & -(\alpha + x_c^2 + \tfrac{1}{4}\,a_1^2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \tag{11}$$

Relation (10) shows two possibilities:

(12) Case 1: $x_c = 0 \quad \rightarrow \quad a_1 = 2\sqrt{-\alpha}$

$$\omega = \sqrt{\beta - 3\alpha}$$

As predicted, $\alpha < 0$ is required for an LC to exist centered about the origin. The second parameter must satisfy $\beta > 3\alpha$, so β can take on any positive value but cannot be more negative than 3α.

(13) Case 2: $x_c = \pm\sqrt{\frac{\beta - 6\alpha}{5}} \quad \rightarrow \quad a_1 = 2\sqrt{\frac{\alpha - \beta}{5}}$

$$\omega = \sqrt{\beta - 3\alpha}$$

For Case 2 limit cycles to exist, it is necessary that $3\alpha < \beta < \alpha$, so again limit cycles cannot exist unless $\alpha < 0$. One additional constraint must be imposed: $|x_c| > a_1$ must hold or the two limit cycles will overlap. This condition reduces the permitted range of β to $2\alpha < \beta < \alpha$.

One final condition should be investigated: for $2\alpha < \beta < \alpha$, the case 2 LC's must lie inside the case 1 LC: from (12) and (13) this is true if

$$\frac{\beta - 6x}{5} + \frac{4(\alpha - \beta)}{5} < -4\alpha$$

$$-2\alpha - 3\beta < -20\alpha$$

which is indeed satisfied over the range $2\alpha < \beta < \alpha$.

The stability of the case 1 LC can be determined as follows: If $a_1^2 = -4\alpha - \varepsilon < -4\alpha$, then F_{DF} is

$$F_{DF} = \begin{bmatrix} 0 & 1 \\ -(\beta - 3\alpha - \frac{3}{4}\varepsilon) & \frac{1}{4}\varepsilon \end{bmatrix}$$

which for $\varepsilon > 0$ has slightly unstable eigenvalues. Thus a trajectory just inside the LC will grow, indicating that the case 1 LC is stable. A similar analysis of the case 2 LC is more complicated, and thus omitted.

Another viewpoint is provided by the traditional singularity analysis approach (refer to [8]), which involves linearization about $x = 0$ and (if $\beta < 0$) $x = \pm\sqrt{-\beta}$. The linearized F-matrices and singularity characterizations for $\alpha < 0$ are given as follows:

$$\underline{x = 0} \quad F = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} \qquad \begin{array}{rl} \beta < 0 & \rightarrow \text{ saddle} \\ 0 < \beta < \frac{1}{4}\alpha^2 & \rightarrow \text{ unstable node} \\ \beta > \frac{1}{4}\alpha^2 & \rightarrow \text{ unstable focus} \end{array}$$

$$\underline{x = \pm\sqrt{-\beta}} \quad F = \begin{bmatrix} 0 & 1 \\ 2\beta & \beta-\alpha \end{bmatrix} \qquad \begin{array}{rl} \beta_1 < \beta < 0 & \rightarrow \text{ unstable node} \\ \alpha < \beta < \beta_1 & \rightarrow \text{ unstable focus} \\ \beta = \alpha & \rightarrow \text{ center} \\ \beta_2 < \beta < \alpha & \rightarrow \text{ stable focus} \\ \beta < \beta_2 & \rightarrow \text{ stable node} \end{array}$$

where

$$\beta_1 = (\alpha - 4) + 2\sqrt{4 - 2\alpha}$$

$$\beta_2 = (\alpha - 4) - 2\sqrt{4 - 2\alpha}$$

The LC analysis and singularity analysis are completely consistent for $\alpha < 0$, $\beta > 2\alpha$. For all $\beta > 0$, the single singularity is unstable, and for $\alpha < \beta < 0$, the three singularities are unstable, so in both cases the predicted existence of a single stable LC is reasonable. For $\beta = \alpha$, the existence of two center singularities at $x_o = \pm\sqrt{-\beta}$ is in exact accordance with the condition $\beta < \alpha$ for two interior limit cycles to exist, with centers $x_c \cong \pm\sqrt{-\beta}$. The only range of β which seems to give rise to contradictory results is $3\alpha < \beta < 2\alpha$, where the disappearance of the two inner LC's is not consistent with the stable nature of the singularities at $x_o = \pm\sqrt{-\beta}$ and the continuing presence of a large stable LC centered about the origin . The seemingly anomalous result that the SIDF analysis predicts the existence of two overlapping LC's for $3\alpha < \beta < 2\alpha$ might suggest that there may in fact be a single "peanut-shaped" LC inside the large stable LC -- but such a conclusion would only be an intuitive speculation. Since the conjectured inner limit cycle would be quite distinctly nonsinusoidal, it would be necessary to include higher harmonics (e.g., $\underline{x} = \underline{x}_c + \text{Re}\,[\underline{a}_1 \exp(i\omega t)] +$

Re $[\underline{a}_3 \exp(3 i\omega t)]$) in the SIDF analysis in order to reveal its presence. Such an assumption gives rise to substantially more complicated LC existence conditions, so it is not pursued here.

In the terminology of bifurcation theory, we observe that the SIDF analysis indicates the following:

- Bifurcation from a single stable singularity at $x = 0$ to a single stable LC centered about $x = 0$ for $\beta > 0$, α passing from positive to negative,
- Bifurcation from one stable LC enclosing three unstable singularities to one stable LC enclosing two unstable LC's and a saddle for $\alpha < 0$, β passing from greater than α to less than α,
- Disappearance of the two inner LC's for $\alpha < 0$, $\beta < 2\alpha$,
- Disappearance of all limit cycles for $\beta < 3\alpha$.

One quite simple analysis has revealed a great deal of the rich variety of behavior that the DE can exhibit.

5. Conclusion. The basic result in Section 2 shows that there are no inherent restrictions to the generality of the SIDF approach to studying limit cycle conditions. Very complicated high-order and highly nonlinear systems of differential equations have been treated using appropriate computer algorithms for solving nonlinear algebraic equations [5]. A second-order example of significant complexity has been treated in detail, illustrating the power of this method.

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