$$\left.\begin{aligned} \alpha_{ij} &\leq -\epsilon < 0 \\ \beta_{ij} &\leq -\epsilon < 0 \\ \sqrt{\alpha_{ij}\beta_{ij}} &= \frac{n-1}{2}(a_{ij}+a_{ji}) \end{aligned}\right\} \quad \begin{aligned} i &= 1, 2, \cdots, n, \forall x, \forall t \\ j &= i+1, i+2, \cdots, n. \end{aligned} \qquad (13)$$

can always be written in the form of (1), although generally not in an unique way. Therefore, 2) can also be considered in the following theorem.

Theorem

Consider the system of differential equations

$$\dot{\mathbf{x}} = A(\mathbf{x}, t)\mathbf{x} \qquad (3)$$

where $\mathbf{x}$ is an n-dimensional vector and A an $n \times n$ matrix whose elements are functions of x and t. Then the system of differential equations (3) is uniformly asymptotically stable in the large about the equilibrium point $\mathbf{x}=0$ if

$$a_{ii} \leq -\epsilon < 0, \qquad i = 1, 2, \cdots, n, \forall \mathbf{x}, \forall t \qquad (4)$$

$$\sqrt{a_{ii}a_{jj}} \geq \frac{n-1}{2}\,|a_{ij}+a_{ji}|, \qquad i = 1, 2, \cdots, n, \forall \mathbf{x}, \forall t \qquad j = i+1, i+2, \cdots, n \qquad (5)$$

where $A(\mathbf{x}, t) = (a_{ij}(\mathbf{x}, t))$.

Proof: Consider as a tentative Liapunov function for the system

$$V = \mathbf{x}'\mathbf{x}. \qquad (6)$$

Then

$$\dot{V} = \mathbf{x}'A^*(\mathbf{x}, t)\mathbf{x} \qquad (7)$$

where

$$A^*(\mathbf{x}, t) = A(\mathbf{x}, t) + A'(\mathbf{x}, t) \qquad (8)$$

and the following expression is obtained on expanding $\mathbf{x}'A^*(\mathbf{x}, t)\mathbf{x}$,

$$\mathbf{x}'A^*\mathbf{x} = \sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(\frac{2}{n-1}a_{ii}x_i^2 + 2(a_{ij}+a_{ji})x_ix_j + \frac{2}{n-1}a_{jj}x_j^2\right). \qquad (9)$$

If

$$a_{ii} \leq -\epsilon < 0, \qquad i = 1, 2, \cdots, n, \forall \mathbf{x}, \forall t \qquad (10)$$

and if

$$\sqrt{a_{ii}a_{jj}} \geq \frac{n-1}{2}\,|a_{ij}+a_{ji}|, \qquad i = 1, 2, \cdots, n, \forall \mathbf{x}, \forall t \qquad j = i+1, i+2, \cdots, n \qquad (11)$$

then $\dot{V}$ may be written

$$\dot{V} \leq -\frac{2}{n-1}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(\sqrt{-\alpha_{ij}x_i} \pm \sqrt{-\beta_{ij}x_j}\right)^2 \qquad (12)$$

where α_{ij} and β_{ij} are defined by the following relation

The negative sign is taken in the bracket if

$$(a_{ij}+a_{ji}) \geq 0, \quad i=1, 2, \cdots, n \quad j=i+1, i+2, \cdots, n \qquad (14)$$

the positive sign is taken in the bracket if

$$(a_{ij}+a_{ji}) < 0, \quad i=1, 2, \cdots, n \quad j=i+1, i+2, \cdots, n \qquad (15)$$

and

$$\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left(\sqrt{-\alpha_{ij}}\,x_i \pm \sqrt{-\beta_{ij}}\,x_j\right)^2 \leq -\epsilon < 0, \quad \forall \mathbf{x}, \forall t. \qquad (16)$$

Thus it has been established that

$$\begin{aligned} V &> 0 && \forall \mathbf{x} \neq 0, \forall t \\ \dot{V} &\leq -\epsilon < 0 && \forall \mathbf{x} \neq 0, \forall t \\ V &\to \infty && \text{as } \|\mathbf{x}\| \to \infty. \end{aligned}$$

Therefore, V is a Liapunov function[1] for the system, and the result is proved.

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Stability of Nonlinear Time-Varying Systems

Introduction

In this correspondence it will be shown that several of the results obtained recently in stability theory can be duplicated or, in some cases, improved upon by using a generalization of the quadratic Liapunov function $V=\frac{1}{2}x^TPx+kx^TMx$. The existence of this class of Liapunov functions was first conjectured to be a necessary and sufficient condition for the asymptotic stability of linear time-invariant systems for all values of a parameter k in the range $0<k<\infty$ by Narendra and Neuman.[1] It was subsequently verified by Thathachar and Srinath.[2] The generalization used here produces rather general statements concerning the stability of nonlinear time-varying systems in terms of the frequency response of the plant, and it parallels much of the work of Narendra and Cho[8],[9] using the functional analysis approach. Some of the results of Zames,[5] Brockett and Forys,[6] and Sandberg[7] can also be shown to be special cases of the results obtained here.

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The Problem

The system considered is described by the vector state equation

$$\dot{x} = Ax - bk(t)f(\sigma), \quad \sigma = h^Tx \qquad (1)$$

This is equivalent to a linear transfer function $G(s)=h^T(sI-A)^{-1}b$ with a single nonlinear time-varying feedback gain $k(t)f(\sigma)$, where σ is the plant output, $0<k(t)<\infty$, $0<f(\sigma)/\sigma<\infty$, and $f(0)=0$. The results can also be extended to cases where $0<k(t)<K_1$, and $0<f(\sigma)/\sigma<K_2$.

It is further understood that the plant is completely controllable, completely observable, and asymptotically stable. The phase variable canonical form can thus be used with no loss in generality[3]

$$h^T = [h_1h_2 \cdots h_n],$$

$$A = \begin{bmatrix} 0 & I \\ -a_1 & -a_2 \cdots -a_n \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \qquad (2)$$

By inspection

$$G(s) = \frac{h_ns^{n-1} + \cdots + h_2s + h_1}{s^n + a_ns^{n-1} + \cdots + a_2s + a_1}. \qquad (3)$$

This transfer function is assumed to have $m<n$ real zeros at $s=-\eta_1, -\eta_2, \cdots, -\eta_m$.

The Liapunov function is of the form

$$V = \frac{1}{2}x^TPx + \sum_{i=0}^{m} k(t)\beta_i\int_0^{\rho_i} f(z)dz,$$

where the matrix P is positive definite ($P>0$), and the signals used in the upper limit are of the form $\rho_i=r_i^Tx$, $\rho_0=\sigma$. Clearly, V is positive definite as long as $B_i>0$.

RL Multiplier: Monotonic Nonlinearities

$$\left[\frac{df(z)}{dz} \geq 0 \quad \text{all } z\right]$$

It has been shown[3] that if

$$r_i^T = \frac{\gamma_i}{\beta_i}h^T(\eta_iI + A)^{-1},$$

then the following relations hold:

a) $r_i^Tb = 0$

b) $r_i^T(sI-A)^{-1}b = \dfrac{\gamma_i}{\beta_i}\cdot\dfrac{G(s)}{s+\eta_i}$

c) $r_i^TAx = \dfrac{1}{\beta_i}(\gamma_i\sigma - \beta_i\eta_i\rho_i)$.

Using these properties it can be seen that

$$\begin{aligned} \dot{V} = {} & \frac{1}{2}x^T(PA + A^TP)x - \beta_0h^Tb[k(t)f(\sigma)]^2 \\ & - \alpha_0k(t)\sigma f(\sigma) - k(t)f(\sigma)\,[Pb - \alpha_0h \\ & - \beta_0A^Th - \sum_{i=1}^{m}\gamma_i(h - r_i)] \\ & - \sum_{i=1}^{m}\gamma_ik(t)(\sigma - \rho_i)[f(\sigma) - f(\rho_i)] \\ & - \sum_{i=1}^{m}k(t)\rho_if(\rho_i)[\beta_i\eta_i - \gamma_i] \\ & + \frac{dk}{dt}\sum_{i=0}^{m}\beta_i\int_0^{\rho_i}f(z)dz. \end{aligned} \qquad (4)$$

This is obtained by adding and subtracting $\alpha_0 k\sigma f(\sigma)$ and

$$\sum_{i=1}^{m} \gamma_i k(\sigma - \rho_i)[f(\sigma) - f(\rho_i)];$$

the first is positive semidefinite for all nonlinearities considered, and the second is positive semidefinite for any monotonic nonlinearity.

Choosing β_i sufficiently large so that $\beta_i\eta_i - \gamma_i = \epsilon_i > 0$ and defining

$$\tau \equiv 2\beta_0 h^T b$$

and

$$F(\rho) \equiv \rho f(\rho) \Big/ \int_0^\rho f(z)dz,$$

the Lefschetz form of the Kalman–Yabubovich lemma can be applied:

Given the stable matrix A, a symmetric matrix $D>0$, vectors $b\neq 0$, and k, and scalars $\tau \geq 0$, $\epsilon > 0$, then a necessary and sufficient condition for the existence of a solution as a matrix P (necessarily >0) and vector q of the system

a) $A^TP + PA = -qq^T - \epsilon D$

b) $Pb - k = \sqrt{\tau}\, q$

is that ϵ be small enough and that the Kalman relation

c) $\tau + 2\mathrm{Re}[k^T(j\omega I - A)^{-1}b] \geq 0$

be satisfied for all ω. This yields

$$\dot V = -\frac{1}{2}[q^Tx + \sqrt{\tau}\, kf(\sigma)]^2 - \epsilon x^T D x - \sum_{i=1}^{m} \gamma_i k(\sigma - \rho_i)[f(\sigma) - f(\rho_i)] - \beta_0 \int_0^\sigma f(z)dz\left[\frac{\alpha_0}{\beta_0} kF(\sigma) - \frac{dk}{dt}\right] - \sum_{i=1}^{m}\beta_i \int_0^{\rho_i} f(z)dz\left[\frac{\epsilon_i}{\beta_i} kF(\rho_i) - \frac{dk}{dt}\right]. \quad (5)$$

The Kalman relation required is

$$\beta_0 h^T b + \mathrm{Re}\left\{\left[\alpha_0 h + \beta_0 A^T h + \sum_{i=1}^{m}\gamma_i(h - r_i)\right]^T (j\omega I - A)^{-1} b\right\} \geq 0. \quad (6)$$

This may be expressed in the form $Z(s)G(s)$ = positive real (p.r.), if the relations

$$\beta_0[h^Tb + h^TA(sI - A)^{-1}b] = \beta_0 sG(s)$$

and

$$\gamma_i(h - r_i)^T(sI - A)^{-1}b = \gamma_i\left(1 - \frac{\gamma_i}{\beta}\frac{1}{s + \eta_i}\right)G(s)$$

are used; [3] they yield

$$Z(s) = (\alpha_0 + \beta_0 s) + \sum_{i=1}^{m} \gamma_i \frac{s + C_i\eta_i}{s + \eta_i} \quad (7)$$

where $0 < C_i = 1 - \gamma_i/\beta_i\eta_i = \epsilon_i/\beta_i\eta_i < 1$. It can be seen that $Z(s)$ has the form of an RL impedance and, hence, is of the form

$$Z_{\mathrm{RL}}(s) = \frac{(s + \lambda_1)(s + \lambda_2)\cdots(s + \lambda_{m+1})}{(s + \eta_1)(s + \eta_2)\cdots(s + \eta_m)} \quad (8)$$

where it is well known that

$$\lambda_1 < \eta_1 < \lambda_2 < \eta_2 \cdots < \eta_m < \lambda_{m+1}.$$

It can be shown[10] that $\gamma_j^* > 0$ always exist such that

$$Z_{\mathrm{RL}}(s) = (s + \lambda_1) + \sum_{j=1}^{m} \gamma_j^* \frac{s + \lambda_1}{s + \eta_j}. \quad (9)$$

Thus in (5) and (7) set

$$\alpha_0/\beta_0 = C_i\eta_i = \epsilon_i/\beta_i = \lambda_1$$

with no loss in generality. Hence the expression for $\dot V$ (5) can be seen to be negative definite for any $k(t)$ satisfying

$$-\infty \leq \frac{dk}{dt} \leq \lambda_1 k F_{\min} \quad (10)$$

where

$$F_{\min} \equiv \min_z \{F(z)\}$$

and λ_1 is the largest value of λ such that $Z(s-\lambda)$ has all its singularities in the left half-plane. Zames[5] obtained a less general result with $F_{\min}$ replaced by unity; for monotonic nonlinearities, $1 < F_{\min} < \infty$ so that the specific form of $f(\sigma)$ may allow a significantly larger upper bound on dk/dt.

Special Cases

1) If a Popov multiplier $(s+\delta_0)$ exists such that $(s+\delta_0)G(s)$ is positive real, then for any nonlinearity (not necessarily monotonic) the requirement is (10) with $\lambda_1 = \delta_0$, where $F_{\min}$ for nonmonotonic nonlinearities has the range $0 < F_{\min} \leq \infty$. For the linear time-varying case $f(z) \propto c$ and $F_{\min} = 2$ so that $\dot k/k \leq 2\delta_0$. This special case has been obtained by Brockett and Forys[6] and Sandberg.[7]

2) If the system is linear time varying and an RL multiplier is used, $dk/dt \leq 2\lambda_1 k$.

RL+RC Multiplier: Odd Monotonic Nonlinearities

$$\left[\frac{df}{dz} \geq 0 \text{ all } z; f(-z) = -f(z)\right]$$

In a development similar to the preceding, a multiplier for odd monotonic nonlinearities of the form

$$Z(s) = (s + \lambda_1) + \sum_{j=1}^{m_1} \gamma_j^* \frac{s + \lambda_1}{s + \eta_j} + \sum_{i=1}^{m_2} \delta_i \frac{s + \mu_i}{s + \eta_i} \quad (11a)$$

is found. The first two terms are the RL portion, and the last term, where $\eta_i < \mu_i < 2\eta_i$, is an RC impedance. This portion is

$$Z_{\mathrm{RC}}(s) = \frac{s + \mu_i}{s + \eta_i} = \frac{s + \left(\eta_i + \frac{\gamma_i}{\beta_i}\right)}{s + \eta_i} = \frac{s + \left(2\eta_i - \frac{\epsilon_i}{\beta_i}\right)}{s + \eta_i}. \quad (11b)$$

If $\delta_{\min} \equiv \min\{\lambda_1, \epsilon/\beta_i\}$ the dk/dt requirement [10] is

$$-\infty \leq \frac{dk}{dt} \leq k\delta_{\min}F_{\min}. \quad (12)$$

A physical interpretation of $\delta_{\min}$ is not apparent at this time. However, this result is still useful and completely new with this correspondence.

For linear time-varying systems, $\dot V$ simplifies to a form which allows

$$\delta_{\min}^* = \min\{\lambda_1, \eta_i\} \quad (13)$$

to replace $\delta_{\min}$ in (12). Note that the η considered are only those that appear in the RC portion of $Z(s)$. Physically, $\delta_{\min}^*$ is the minimum of the zeros of the RL portion of the multiplier and the poles of the RC term.

Conclusion

The authors feel it should be evident that the form of the Liapunov function used has proved to be of significant value in providing simple stability criteria in a number of problems, which have been solved in other ways, as well as in the new case of odd monotonic nonlinearities. For further details the reader is referred to Narendra and Taylor.[10]

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A Graphical Method for Finding the Real Roots of nth-Order Polynomials

Introduction

The problem of determining the roots of a polynomial is fundamental to many fields of engineering, and much effort has been expended in this direction. Since it is not possi-

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