Comment on "Volterra Series Synthesis of Nonlinear Stochastic Tracking Systems"

Abstract

The superiority of the Covariance Analysis Describing Function Technique (CADET) in comparison with the Volterra Series Synthesis approach for the analysis of nonlinear stochastic systems is demonstrated in this note. The advantages of CADET are greater accuracy and simplicity.

The purpose of this correspondence is to describe an alternative analytic technique to that presented by Landau and Leondes in the above paper [1] which is both simpler to apply to nonlinear systems with random inputs and more accurate. The alternative technique is the covariance analysis describing function technique (CADET™)[1] which is outlined in Gelb [2] and surveyed most recently in Taylor et al. [3]. In essence, each system nonlinearity is quasilinearized, resulting in a coupled, nonlinear set of differential equations for propagating the second-order state variable statistics (mean vector **m** and covariance matrix S) as functions of time, from specified initial conditions $\mathbf{m}_0$, S_0. The technique can be readily applied to state equations of any order, with any number of nonlinearities; it has proven to be about as accurate as Monte Carlo analysis using 200 trials, but much more efficient in terms of computer time expenditure.

To study the accuracy of CADET vis-à-vis the second-order approximate solution of Landau and Leondes based on Volterra series expansions (hereafter called the second-order Volterra solution), a few cases treated in [1] were analyzed using CADET, and Monte Carlo simulations were performed to provide a basis for comparison.

In summary, the analysis concerns a second-order nonlinear model of a radar tracking system, shown in Fig. 1 in block diagram form, with a receiver characteristic of the form

$$f(e) = e - k_a e^3; \qquad k_a = 0.4 \text{ deg}^{-2}$$

operating on the tracking error e. The tracking error is subject to random initial conditions which are assumed to be Gaussian, specified by the second-order statistics m_{e0}, σ_{e0}. Receiver noise n is modeled as a zero-mean Gaussian white noise process with a power spectral density of amplitude N_0, added after the receiver characteristic. The goal of the analysis is to determine tracking capability for targets described by a unit step line-of-sight (LOS) angle rate,

$$\dot{\sigma}(t) = \Omega u_{-1}(t)$$

where σ is the target LOS angle in an inertial frame and u_{-1} denotes the unit step function.

Before passing to the "random cases" of Landau and Leondes, observe that the "nonrandom case" initially considered in [1] (no receiver noise and deterministic tracking error initial conditions) can be analyzed exactly using CADET. Thus the CADET solutions for the examples shown in [1, Figs. 4 through 9] correspond to the curves labeled "computer solution," while the second-order Volterra solutions are not particularly accurate when a target with a rapidly varying LOS angle (Ω = 6 deg/s) is being tracked.

The random cases shown here correspond to [1, Figs. 10 and 11]; to summarize the conditions, m_{e0} = 0.4 deg, σ_{e0} = 0.1 deg, N_0 = 0.004 deg^2/Hz, and Ω = 4 and 6 deg/s. In the 4-deg/s case, depicted in Fig. 2, the 1000-trial Monte Carlo analysis[2] demonstrates that the CADET result is

[1] CADET is a trademark of The Analytic Sciences Corporation.

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[2] Associated with each Monte Carlo estimate of m_e and σ_e is a confidence band, indicating the range within which the true statistics lie with probability 0.95; they are based on estimated higher-order statistics, not chi-square analysis. This approach is based on the statistics of the sample mean and sample variance for general distributions given in [4].

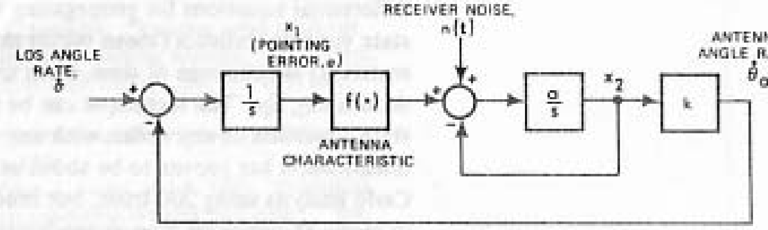
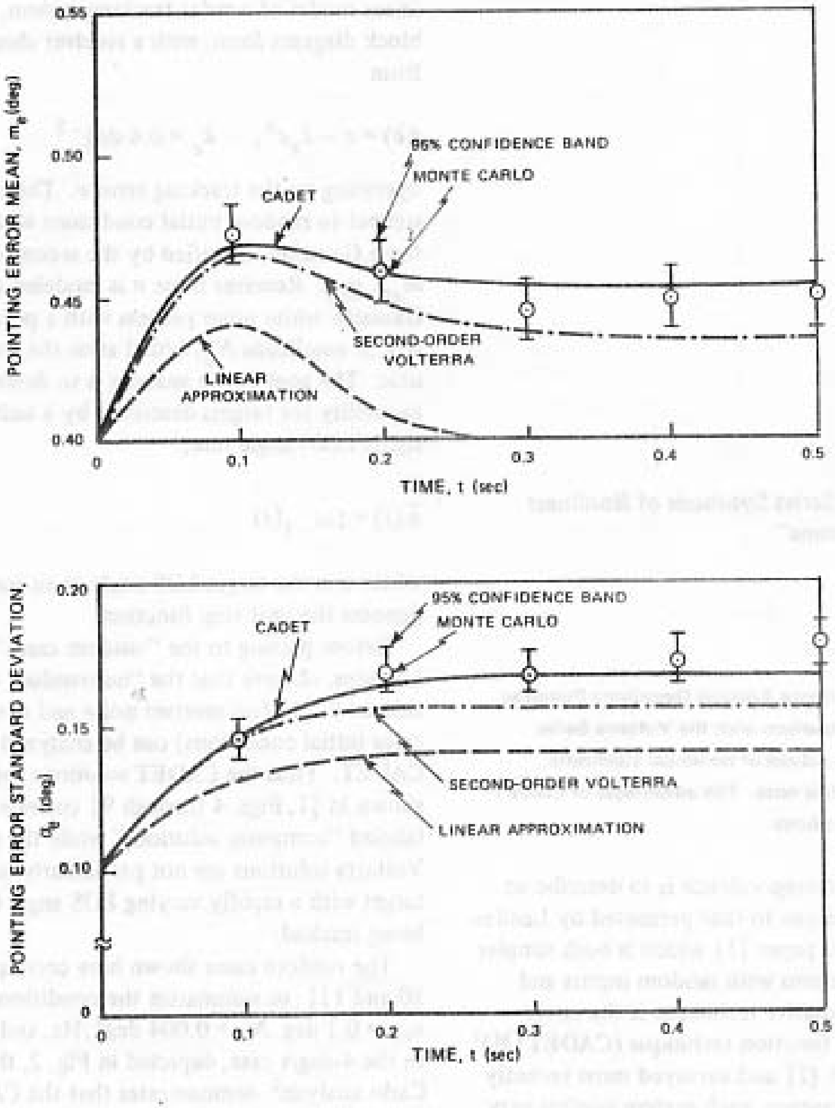


Fig. 1. Antenna pointing and tracking model.

Fig. 2. Pointing error statistics for Ω = 4 deg/s.



very probably more accurate than the second-order Volterra series analysis, although the difference between the CADET and second-order Volterra series results is not large (4 percent for the mean, 6.6 percent for the standard deviation).

The superiority of CADET is more apparent for Ω = 6 deg/s, as shown in Fig. 3. The second-order Volterra series results *do not capture the instability* that results when there is a significant probability that $|e| > 0.91$ deg, i.e., e lies in the region where df/de is negative. The CADET analysis predicts the unstable growth in e quite accurately for the first part of the simulation; after 0.2 s the process is so

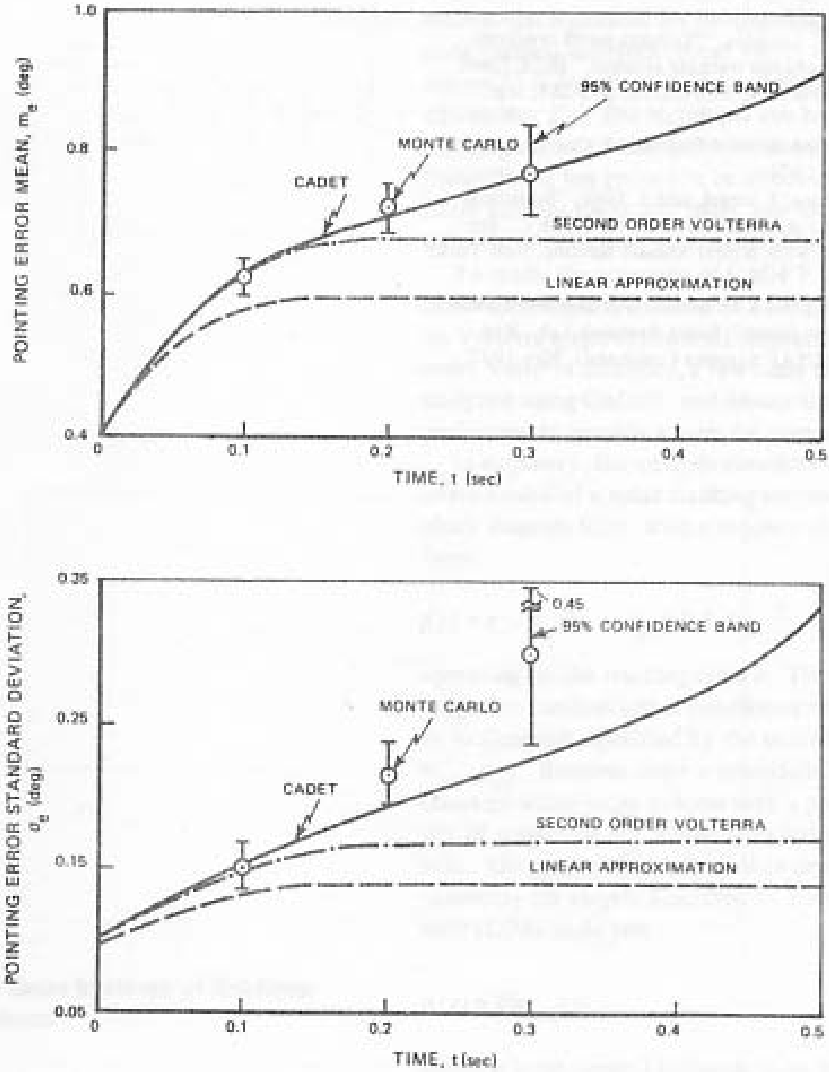


Fig. 3. Pointing error statistics for $\Omega = 6$ deg/s.

TABLE I
Summary of CADET Equations

State space formulation, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{w}$:

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} -kx_2 \\ a[f(x_1) - x_2] \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} \dot{\theta}_t \\ an(t) \end{bmatrix}$$

CADET mean equation, $\dot{\mathbf{m}} = \hat{\mathbf{f}} + \mathbf{b}$:

$$\hat{\mathbf{f}} = \begin{bmatrix} -km_2 \\ a\{m_1[1 - k_a(m_1^2 + 3s_{1,1})] - m_2\} \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} \Omega \\ 0 \end{bmatrix}$$

CADET covariance equation, $\dot{S} = NS + SN^{\mathrm{T}} + Q$:

$$N = \begin{bmatrix} 0 & -k \\ a[1 - 3k_a(m_1^2 + s_{1,1})] & -a \end{bmatrix}; \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & a^2 N_0 \end{bmatrix}$$

Note: The scalars m_i and $s_{i,j}$ are elements of $\mathbf{m}$ and S.

highly non-Gaussian that it is difficult to assess the accuracy of the results (CADET or 200-trial Monte Carlo) without performing many more trials.

The simplicity of CADET application is demonstrated in [3], where the differential equations for $\mathbf{m}$ and S for the problem considered here are developed as an illustration. These equations, summarized in Table I, provide a contrast to those presented in [1]. For the sake of brevity, the latter are not included in this correspondence.

In summary, the advantages of CADET over the second-order Volterra series approach—accuracy and simplicity—are clearly established in the examples shown here. Further details on CADET may be found in [2] and [3].

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