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A SYSTEMATIC NONLINEAR CONTROLLER DESIGN APPROACH BASED ON QUASILINEAR SYSTEM MODELS

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Abstract

Recent research in the area of describing function approaches (both random-input and sinusoidal-input) has laid the foundation for a systematic approach to designing controllers for nonlinear plants. This paper provides an overview of one school of thought in this area, typified by the following key ideas:

1. *Quasilinear* models of the nonlinear system that account for the *operating range* of system variables must be more realistic than conventional linear models.
2. A *staged design approach* that applies more power to "difficult" nonlinear plants than to "easy" ones is more sensible than a single design method.
3. The designer should be led to a *nonlinear* controller design if and only if it is really required.

These points will be developed, illustrated and justified in this presentation.

Introduction

One of the most vexing problems in control system design is the gap between the elegant methods of linear control theory and "real world control problems." A major cause for this gap is that plant nonlinearity is often severe enough that a standard linear approach to control system design cannot be made to work satisfactorily without a substantial amount of cut-and-try modification and "tuning." Some help is available in the results of absolute stability theory[1,2] and robustness criteria;[3] performance considerations often make these results unusable, however, since the stability or robustness conditions are generally very conservative, in the sense that too much performance has to be sacrificed in order to guarantee stability or robustness in the face of nonlinearity. Overall, this situation is not a happy state of affairs for most control design practitioners.

An approach that offers a great deal of promise in dealing with this problem is quasilinearization or the *describing function* (DF) technique. It has not received a great deal of attention in the Western literature in recent years, for several reasons: First, most of the original work in DF theory (refer to Atherton[4] or Gelb and Vander Velde[5] for an overview) dealt with the classical "single nonlinearity in the feedback path" problem, which does not provide much help in dealing with modern real-world problems. Second, most DF approaches were formulated in the frequency domain, which has received little attention in the last two decades until quite recently. Finally, DF methods have been depreciated by many because they are inexact.

The first of these objections has been answered in the last decade. Both the sinusoidal-input and the random-input DF methods have been extended so that they can deal with plants that have any number of nonlinearities, in any configuration.[6,7] This generalization includes even systems having multi-input nonlinearities. The second barrier is also being removed, as control theorists are coming to recognize that the frequency domain is a very powerful arena for dealing with robustness,[3] unmodeled dynamics,[8] control system design,[9,10] and other considerations. Finally, while the DF method is inherently inexact, there is growing recognition that DF's have capabilities for the analysis and design of systems that is simply not available in any other approach.

The goal of this presentation is to show how one can use DF methods as a basis for a systematic approach to designing controllers for nonlinear plants. We will proceed as follows:

1. A review of fundamental DF concepts
2. An overview of DF use in control system design
3. Two methods for designing linear controllers
4. A method for designing nonlinear controllers
5. Analogous concepts using random-input DF's
6. Summary and conclusions

Fundamental DF Concepts

The basic idea of the describing function (DF) approach for studying and modeling nonlinear system behavior is to replace each system nonlinearity with a (quasi)linear term whose "gain" is a function of "input amplitude," where the form of input signal is assumed in advance. This technique is dealt with very thoroughly in a number of texts;[4,5] it may be summarized, in the primary context of the proposed methods for control design for nonlinear plants, as follows:

The nonlinear plant under consideration is characterized by the general state-variable differential equation and output equation

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}), \quad \underline{y} = \underline{h}(\underline{x}, \underline{u}) \tag{1}$$

where $\underline{x}$ is an n-dimensional state vector, $\underline{u}$ is an m-dimensional input vector, and $\underline{y}$ is a p-dimensional output vector. We are going to concern ourselves with the behavior of the plant in the presence of sinusoidal signals, for reasons that are given below, so we take $\underline{u}$ to have the form

$$\underline{u}(t) = \underline{u}_o + \mathrm{Re}[\underline{a}\exp(j\omega t)] \tag{2}$$

where $\underline{u}_o$ is a real vector denoting an operating point of interest, and $\underline{a}$ is a complex-valued vector designating the sinusoidal component amplitude and phase in the standard phasor notation. In accordance with the usual DF assumption, the state variables are assumed to be nearly sinusoidal,

$$\underline{x}(t) \approx \underline{x}_c + \mathrm{Re}[\underline{b}\exp(j\omega t)] \tag{3}$$

where $\underline{b}$ is a *complex amplitude vector* and $\underline{x}_c$ is the state vector *center value* (which is not a singularity, or solution to $\underline{f}(\underline{x}_o, \underline{u}_o) = \underline{0}$, unless the nonlinearities satisfy certain stringent symmetry conditions with respect to $\underline{x}_o$, in which case $\underline{x}_o$ and $\underline{x}_c$ are identical). Then we neglect higher harmonics, to make the approximations

$$\begin{aligned} f(\underline{x}, \underline{u}) \approx\ & \underline{f}_B(\underline{u}_o, \underline{a}, \underline{x}_c, \underline{b}) \\ & + \mathrm{Re}[A(\underline{u}_o, \underline{a}, \underline{x}_c, \underline{b}) \cdot \underline{b}\sin\omega t] \\ & + \mathrm{Re}[B(\underline{u}_o, \underline{a}, \underline{x}_c, \underline{b}) \cdot \underline{a}\sin\omega t] \\ h(\underline{x}, \underline{u}) \approx\ & \underline{h}_B(\underline{u}_o, \underline{a}, \underline{x}_c, \underline{b}) \\ & + \mathrm{Re}[C(\underline{u}_o, \underline{a}, \underline{x}_c, \underline{b}) \cdot \underline{b}\sin\omega t] \\ & + \mathrm{Re}[D(\underline{u}_o, \underline{a}, \underline{x}_c, \underline{b}) \cdot \underline{a}\sin\omega t] \end{aligned} \tag{4}$$

Minimum mean square approximation error is achieved when the real vectors $\underline{f}_B$ and $\underline{h}_B$ and the matrix set $\{A, B, C, D\}$ are obtained by taking the first two terms of the Fourier expansions of the elements of $\underline{f}(\underline{x}_c + \text{Re}[\underline{b}\exp(j\omega t)], \ \underline{u}_o + \text{Re}[\underline{a}\exp(j\omega t)])$. This approach has been illustrated in detail.[4,5,7,11,12] The DF arrays $\{\underline{f}_B, \underline{h}_B\}$ and $\{A, B, C, D\}$ in Eqn. (4) provide the quasilinear representation of the nonlinear plant in Eqn. (1). Observe that the constant or d.c. portion of the model is embodied in $\underline{f}_B$ and $\underline{h}_B$, while the matrices $\{A, B, C, D\}$, which conform to the usual linearized model notation, characterize the plant response to sinusoidal inputs. The two signal components (d.c., first harmonic) are coupled, as the above notation suggests, due to the failure of superposition in nonlinear systems. To stress the fact that this model deals with periodic behavior, we will henceforth refer to this model as an SIDF (sinusoidal-input DF) model of the system.

A wide variety of SIDF's have been catalogued.[4,5] In addition, there is another, more direct way to obtain a DF-like amplitude-sensitive plant model: Simulation plus fast fourier transform methods.[19] Using either approach, the designer can obtain the required SIDF model in a straightforward manner. Thus, we will not consider the computational aspects of the DF approach further.

Using DF Models in Controller Design

Once an SIDF representation of a nonlinear plant is obtained, it may be used in several ways. The most traditional SIDF analysis problem is seeking limit cycle conditions.[4,5,7,11,12] A modern algebraic approach to limit cycle analysis was first fully developed by the author,[13] and is treated in detail in elsewhere.[7,11,12]

Determining the approximate response of a nonlinear plant to sinusoidal inputs follows an approach similar to that used in limit cycle analysis. Applying the same conditions of harmonic balance that underlie limit cycle analysis using the DF approach,[4,5,7] it is possible to solve for $\{\underline{x}_c, \underline{b}\}$ determined by $\{\underline{u}_o, \underline{a}\}$ using

$$\underline{f}_B(\underline{u}_o, \underline{a}, \underline{x}_c, \underline{b}) = \underline{0}$$
$$\underline{b} = (j\omega I - A)^{-1} B \, \underline{a} \tag{5}$$

These 2n nonlinear algebraic equations (n of which are complex-valued) can be solved readily using standard computer routines. In this case, one should be careful to ensure that A does *not* have eigenvalues on or very close to the imaginary axis; otherwise limit cycles may exist in addition to the response to the sinusoidal input, in contradiction to the assumptions underlying Eqns. (3 - 5).

The sinusoidal component of the plant response can then be characterized by the input-amplitude-dependent matrix "transfer function"

$$G(j\omega; \underline{u}_o, \underline{a}) = C(j\omega I - A)^{-1} B + D \tag{6}$$

where we observe that the SIDF matrices are explicitly determined by $\{\underline{u}_o, \underline{a}\}$, since $\{\underline{x}_c, \underline{b}\}$ are eliminated using Eqn. (5).

This modern algebraic method for ascertaining the $\{A, B, C, D\}$ set and "frequency response" (Eqn. (6)) for a nonlinear plant was proposed in Taylor,[14] and serves as the basis for the controller design techniques delineated in this paper.

Based on the above mathematical development, we can summarize a number of considerations that motivate the use of SIDF models rather than conventional linear models as the basis for controller design:

1. SIDF models are *amplitude dependent,* i.e., they account for and realistically characterize the input/output behavior of a nonlinear element or plant over a range of input variation (specified by the analyst or designer) rather than merely for "small" (infinitesimal) variation.
2. SIDF models can deal with *discontinuous* and *multi-valued* nonlinear devices (e.g., relays, "stiction" or static friction, relays with hysteresis, backlash), while conventional linearization cannot.
3. The SIDF model for small input variational ranges approaches the conventional linear model (provided the nonlinearities are differentiable and single-valued, so that such a model exists).
4. The departure of individual SIDF gains from their small-variation values provides a useful *measure of the relative impact of a particular nonlinear element* under the specified operating conditions.
5. The variation of the SIDF model (especially, variation of its eigenvalues, eigenvectors and singular values) with respect to changes in input amplitude $\underline{a}$ provides a good *measure of model sensitivity.* Such a measure is completely lacking in conventional linear models.
6. General SIDF models are not particularly difficult to obtain, either by analytic means[4,5,7,11,12] or by simulation combined with fast Fourier transform techniques.

From point 3, we observe that the SIDF model subsumes the conventional linear model, when the latter exists, so there need be little concern that the designer will obtain "strange" results. The facts that SIDF models are more realistic, that they allow one to identify those nonlinear effects that have an impact on plant behavior for a given set of operating conditions, and that they may be perturbed with respect to input amplitude to determine sensitivity are of major significance in the control design context. In addition, the *usefulness* of general DF methods lies in the subsequent treatment of the resulting quasilinear model using linear system analysis and design techniques, which are well established and usually very straightforward to apply.

One issue that is generally not a major concern is the sensitivity of DF's to the exact form of the nonlinearity input signals, here taken to be sinusoidal. The general concept of the DF can be extended to treat signals of any amplitude distribution (cf. Gelb and Vander Velde[5]), and the easiest way to demonstrate that signal form sensitivity is not a big issue is to calculate DF's for a common nonlinearity under a number of assumed distributions, and see how much variation is obtained in the corresponding DF gain versus input amplitude plots. An example is shown in Fig. 1, where it can be observed that for a rather comprehensive set of input amplitude distributions (gaussian, triangular, uniform, sinusoidal, and bimodal triangular) the gain is not very sensitive: The gains for a limiter with these input distributions are within 10 to 15 percent of the gain plot for uniformly distributed inputs, which is about as accurately as the designer will know any model parameter.

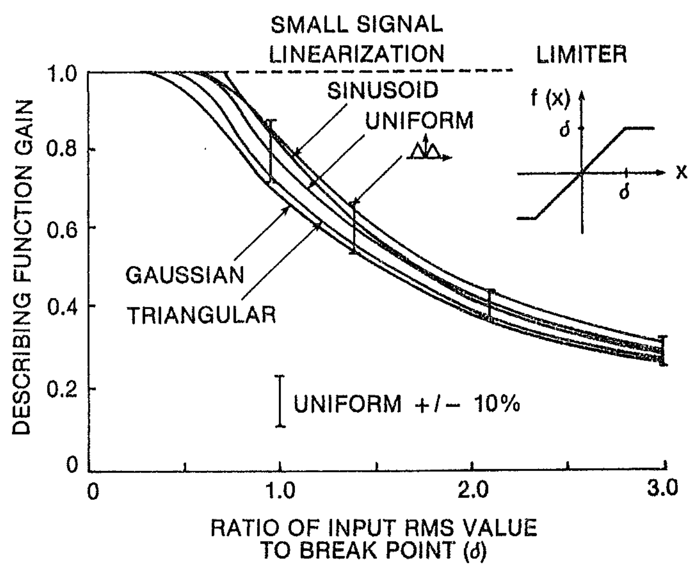

Figure 1. DF Gain Sensitivity to Input Distribution

It is necessary, however, to reiterate the standard caveat regarding the use of DF models for nonlinear systems: Although DF models are more realistic than those based on conventional linearization in many cases, it is always possible to find pathological systems where DF models are not useful. Generally speaking, these are cases where higher harmonic effects are dominant (either due to high-frequency resonances or to particularly "strong" nonlinear effects). The use of simulation as an adjunct to DF methods to validate assumptions and final system designs is thus always advised.

The general suggestion that DF models can be used to advantage in control system design is of long standing. Standard DF texts[4,5] treat this approach in detail, and provide references for its inception. One of the major reasons why this idea may not have enjoyed widespread use appears to be the lack of generality of earlier DF model formulations. In particular, the standard system configuration was formerly restricted to plants modeled by a linear scalar transfer function of n^{th} order in series with a memoryless, single-input/single-output (SISO) nonlinear element. Generalization of this configuration proved to be a major barrier, especially in the area of SIDF theory. Until recently, such extensions were typically very restrictive, e.g., nonlinear systems could be comprised of a series interconnection of alternating linear dynamic blocks followed by SISO nonlinearities, or of a parallel network of <linear dynamics plus SISO nonlinearity> paths. The first extension of DF theory into the general state-space model format of Eqn. (1) was in the realm of random-input DF's (RIDF's; see Kazakov,[15] Gelb and Warren[16]). More will be said about this approach below. For the case of sinusoidal-input DF's, Eqn. (4), early contributions are Taylor[13] and Hannebrink, et al.[17]

The recent use of SIDF models as a basis for controller design in the context of multi-variable systems has been actively pursued by Gray.[18] The approach developed therein is a major step forward compared with earlier SIDF methods, in terms of system configuration constraints; it is not applicable to the completely general plant formulation in Eqn. (1), however. Also, it does not take full advantage of the computational simplicity and efficiency of the modern algebraic solution outlined in this paper. Finally, it is based on the concept of limit cycle avoidance, which does not address performance issues or any other aspects related to the concept of *design operating range* which is employed here.

Linear Controller Design Methods

Single-Range Linear Controller Design

The basic ideas of this lowest-level SIDF-based controller design approach are as follows:

1. The designer supplies a state space nonlinear model, as in Eqn. (1), and an operating range $\{\underline{u}_o, \underline{a}\}$ in Eqn. (2).
2. DF analysis or FFT methods are used to obtain the complete state vector amplitudes $\{\underline{x}_c, \underline{b}\}$ determined by $\{\underline{u}_o, \underline{a}\}$ and the quasilinear model in the form $\{A, B, C, D\}$ or $G(j\omega;\, \underline{u}_o, \underline{a})$.
3. A conventional linear design method is used to obtain a controller for the SIDF model that meets desired performance specifications.
4. The performance of the controller is verified by simulation.

This basic SIDF approach to linear controller design is discussed in greater detail elsewhere.[19] In that paper, many issues regarding the practical implementation of this idea and of SIDF methods in general are treated in somewhat greater depth.

Two-Range Linear Controller Design

A linear controller based on a SIDF model for a single operating range may not perform adequately for small signals (small compared with the specified design range), or it might not perform well for larger signals (especially, large signals may cause the closed-loop system to become unstable). It may still be possible to obtain a linear controller that will not suffer these deficiencies. The following procedure should allow the designer to obtain such a controller with a minimal amount of trial-and-error:

1. Obtain a second model for "small" or "large" signals, as appropriate to the problem. (A small-signal model may be a conventional linear model if the nonlinearities are differentiable; otherwise, one may use another SIDF-based model based on a value $\underline{a}'$ that is appropriately small compared with the original $\underline{a}$.) Denote this model $G'(j\omega;\, \underline{u}_o, \underline{a}')$.
2. Fit the two SIDF models with stable rational matrix factorized representations.[9,10]
3. Proceed with linear controller design using the simultaneous stabilization approach of Vidyasagar and Viswanadham[20] to obtain a characterization of *all* linear controllers that will simultaneously place the closed-loop poles of G and G′ (the poles of a feedback system with G or G′ compensated by such a controller) in a desired portion of the s-plane.
4. Use an optimization procedure to get the *best* linear controller, i.e., the member of the set found in 3. that most nearly meets the designer's performance specifications for the nominal DF-based plant model $G(j\omega;\, \underline{u}_o, \underline{a})$.
5. Determine if the 'best' controller achieves satisfactory performance. If so, accept it; if not, design a nonlinear controller (see below).

The rationale for this procedure is as follows: The problems that the designer will most often encounter in trying to use the single-range SIDF-model-based method outlined in the beginning of this section will be that

1. The operating point of the resulting closed-loop system will not be stable (generally, there will thus exist small limit cycles), or
2. Signals somewhat larger in amplitude than the design range will cause instability.

These problems are most likely to occur when the nominal-signal SIDF plant model is substantially different from that obtained for smaller or larger signals. The new approach outlined above will force the designer to restrict himself to the class of linear controllers that will deal with that problem. The controller design is completed by getting as close as possible to the desired performance for the nominal operating range; if this fails to achieve satisfactory behavior, then the designer will be forced to resort to Inverse DF Synthesis (see below) of a nonlinear controller.

Designing Nonlinear Controllers

The two methods outlined above should provide the designer with the tools required to arrive at an acceptable linear controller design in many situations where the plant is not extremely nonlinear. If the plant nonlinear effects are less benign, however, even simultaneous stabilization based on two SIDF models may not produce adequate results. The following approach, which we will call the *Inverse DF Synthesis Method* should prove to be effective in many such situations:

1. Choose a *number of design ranges* as defined by $\{\underline{u}_o, \underline{a}\}$ in Eqn. (2) (at least three ranges are needed, e.g., small, medium, large), and obtain a SIDF model for each.
2. Select a *fixed controller configuration* (e.g., PID, lag, lag-lead, etc.).
3. Design a controller of the selected type *for each SIDF model* (controllers must have a common configuration; some or all controller parameters will differ for different design ranges).
4. Interpret the varying controller parameters as *describing functions* for controller nonlinearities; invert the SIDF's to obtain the corresponding nonlinear element.

An illustration of this approach is provided in Fig. 2: In part (a), the common PID controller gains are plotted as functions of the appropriate controller signal amplitude (e, integral of e, or derivative of e). One can see that the "gain patterns" make the choice of a corresponding nonlinearity readily apparent, as shown. By fitting the corresponding SIDF to each gain pattern, the nonlinear

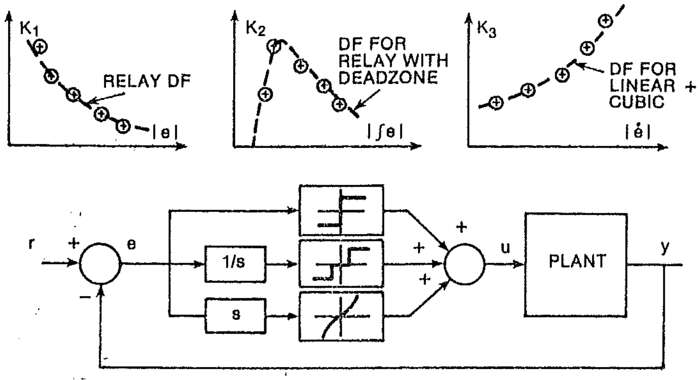

Figure 2. Illustration of the Inverse DF Synthesis Method

controller in part (b) is easily obtained. Of course, there are many nonlinear characteristics that will yield a given pattern; the designer is free to choose any one of these according to its ease of implementation, or any other consideration.

There are several ways one might carry out the SIDF inversion process needed for the last step. If the number of ranges considered is small, e.g., three, then the number of possible gain patterns is small. For three ranges there are four patterns: monotonically decreasing (K1), monotonically increasing (K3), concave up (K2), and concave down. One would only require a "catalog" of four nonlinearity types, and a perfect gain/SIDF fit can be achieved. If more ranges must be considered, then the use of a general piece-wise linear characteristic, Fig. 3, can be recommended, where the number of segments, slopes, and breakpoints can be selected to fit any gain pattern.

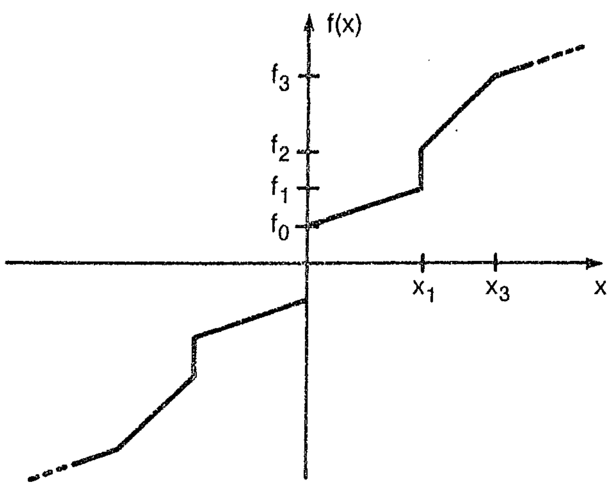

Figure 3. A General Piecewise-Linear Nonlinearity

The basic idea that underlies this procedure is that the original linear controller designs that are to be coalesced into the final nonlinear controller by SIDF inversion must be based on the same basic performance objectives or design specifications. The nonlinear closed-loop system obtained by this approach will then be "nearly uniform" in terms of the performance measures used in designing the controllers for the set of signal ranges (e.g., small, medium, large). If it is not possible to design for a common specification for the set of SIDF models under consideration, then the designer must try another controller configuration; otherwise the Inverse SIDF Synthesis step does not make sense and will probably not succeed.

The justification for this third design technique is simply that a nonlinear controller is much more likely to provide the desired performance over a wide range of operation than a linear one if the basic input/output behavior of the plant (as characterized by the various SIDF models) differs substantially. This new approach to nonlinear controller design is completely systematic, and the designer is not trying to directly cancel nonlinearities, which is at best a questionable practice and is often not possible or practical.

It is the author's opinion that this method promises to be a major breakthrough, in terms of being able to deal with difficult nonlinear problems. It is always to be recognized that no single approach or set of approaches will handle every conceivable situation — but the above three-stage attack should prove to be quite powerful.

Analogous Ideas for Random-Input DF's

The use of RIDF models derived from the general state-space plant formulation, Eqn. (1), as a basis for control system design has been suggested by Hedrick.[21] Such an approach is especially appropriate in the context of nonlinear stochastic systems, as the resulting RIDF model can be used directly as the basis for an LQG (linear-quadratic-gaussian) controller design. There are several issues that would appear to explain the lack of common use of this statistical linearization approach, particularly in industrial applications:

1. It is not widely known that RIDF models are readily obtainable for general system configurations (cf. Taylor et al.[6] for an overview of applicable RIDF results for non-SISO nonlinear elements).
2. It is not possible to deal with multi-valued nonlinear effects (hysteresis, backlash) in a meaningful manner.
3. Many designers are not familiar with the stochastic control problem, and work more easily in the frequency domain.
4. It is not possible to use RIDF models to investigate possible limit cycle conditions, while the SIDF approach is powerful in this arena.[7] The occurrence of limit cycle behavior is a very common cause of "unsatisfactory performance."

The use of SIDF's is thus a more natural and powerful approach, in light of the above points.

It is worth pointing out that the SIDF-based design approaches outlined in previous sections can be carried over to the RIDF technique in a very direct manner. In particular:

1. The single-range DF model as a basis for linear controller design is already known.[21]
2. The concept of using simultaneous stabilization[20] on RIDF models for two operating ranges carries over directly. Two $\{A, B, C, D\}$ sets may be obtained for different input spectral density values, then these RIDF models can be converted into frequency domain models, and simultaneous stabilization (or designing a controller to place the closed-loop system poles in a desired region of the s-plane) can be carried out as outlined above.
3. The multi-range approach wherein the designer obtains DF models for three or more operating ranges, designs linear controllers of the same configuration and with the same design objectives, then uses DF inversion to coalesce the linear designs into a single nonlinear controller also presents no conceptual difficulty.

Applying these ideas would be a valuable research effort.

Summary and Conclusions

The modern algebraic formulation of the sinusoidal-input describing function method for modeling and analyzing nonlinear systems brings this approach to such a state of power, generality, and analytic simplicity that it should come to play a major role in nonlinear systems design. The design approaches outlined in this paper should provide a good framework for such a development.

References

[1] Narendra, K.S. and Taylor, J.H., *Frequency Domain Criteria for Absolute Stability,* Academic Press, New York, 1973.

[2] Taylor, J.H., "Applications of Rigorous Stability Criteria to Nonlinear Dynamical Systems," Ch. 20 of *Nonlinear System Analysis and Synthesis: Vol. 2 — Techniques and Applications,* Ed. by Ramnath, R.V., Hedrick, J.K., and Paynter, H.M., the Amer. Soc. of Mech. Engrs., 1980.

[3] Vidyasagar, M., "The Graph Metric for Unstable Plants and Robustness Estimates for Feedback Stability," Report # 82-04, University of Waterloo (Accepted for publication in IEEE *Trans. on Automatic Control*).

[4] Atherton, D.P., *Nonlinear Control Engineering,* Van Nostrand Reinhold, London, 1975 (Student Edition, paperback, 1982).

[5] Gelb, A. and Vander Velde, W.E., *Multiple-Input Describing functions and Nonlinear System Design,* McGraw-Hill, New York, 1968.

[6] Taylor, J.H., Price, C.F., Siegel, J. and Gelb, A., "Covariance Analysis of Nonlinear Stochastic Systems," Ch. 13 of Ref. 2 above, 1980.

[7] Taylor, J.H., "Applications of a General Limit Cycle Analysis Method for Multivariable Systems," Ch. 9 of Ref. 2 above, 1980.

[8] Doyle, J.C., and Stein, G., "Multivariable Feedback Design: Concepts for a Classical/Modern Synthesis," IEEE *Trans. on Automatic Control,* Vol. AC-26, pp. 4-16, February, 1982.

[9] Callier, F.M. and Desoer, C.A., *Multivariable Feedback Systems,* Springer-Verlag, 1982.

[10] Vidyasagar, M., *Control System Synthesis: A Factorization Approach,* Lecture Notes, to be published by MIT Press.

[11] Taylor, J.H., "Describing Function Method for Limit Cycle Analysis of Highly Nonlinear Systems," IX Int. Conf. Nonlinear Oscillations, Kiev, USSR, 1981.

[12] Taylor, J.H., "General Describing Function Method for Systems With Many Nonlinearities, With Application to Aircraft Performance," Joint Auto. Control. Conf., San Francisco, 1980.

[13] Taylor, J.H., "An Algorithmic State-Space/Describing Function Technique for Limit Cycle Analysis," TIM-612-1, The Analytic Sciences Corp. (TASC), Reading, MA, 1975.

[14] Taylor, J.H., "Describing Function Methods for High-Order Highly Nonlinear Systems," Int. Cong. on Appl. Systems Research and Cybernetics, Acapulco, Mexico, 1980.

[15] Kazakov, I.E., "Generalization of the Method of Statistical Linearization to Multidimensional Systems," *Avtom. i Telemekh.,* Vol. 26, 1210-1215, 1965.

[16] Gelb, A. and Warren, R.S., "Direct Statistical Analysis of Nonlinear Systems: CADET," *AIAA J.,* Vol. 11, 689-694, 1973.

[17] Hannebrink, D.N., Lee, H.S.H., Weinstock, H. and Hedrick, J.K., "Influence of Axle Load, Track Gauge, and Wheel Profile on Rail Vehicle Hunting," *Trans. ASME-J. Eng. Ind.,* 186-195, 1977.

[18] Gray, J.O., and Taylor, P.M, "Computer-Aided Design of Multivariable Nonlinear Control Systems Using Frequency Domain Techniques," *Automatica 15,* 281-297, 1979.

[19] Taylor, J.H., "Robust Computer-Aided Control System Design for Nonlinear Plants," Application of Multivariable Systems Theory, Manadon, Plymouth, UK, October 1982.

[20] Vidyasagar, M. and Viswanadham, N., "Algebraic Design Techniques for Reliable Stabilization," IEEE *Trans. on Automatic Control,* Vol. AC-27, October 1982.

[21] Hedrick, J.K., "The Use of Statistical Describing Functions with Linear-Quadratic-Gaussian Controller Design," Joint Auto. Control. Conf., West Lafayette, IN, pp. 390-393, 1976.