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SYNTHESIS OF NONLINEAR CONTROLLERS WITH RATE FEEDBACK VIA SINUSOIDAL-INPUT DESCRIBING FUNCTION METHODS

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Abstract

In this paper, we report on recent advances in the design of fully nonlinear controllers for amplitude-sensitive nonlinear plants using sinusoidal-input describing function methods. This work includes the development of a new nonlinear controller synthesis approach that includes derivative action in an inner-loop feedback path (nonlinear rate feedback), and its application to a motor + load model with torque saturation and stiction. This approach is capable of treating nonlinear systems of a very general nature, with no restrictions as to system order, number of nonlinearities, configuration, or nonlinearity type; additionally, the techniques can be generalized for the design of nonlinear controllers of different structures. The end result is a closed-loop nonlinear control system that is relatively insensitive to reference-input amplitude.

1. INTRODUCTION

This paper treats recent work in the development of nonlinear control system design techniques based on sinusoidal-input describing function (SIDF) methods. The basis of this work has been established previously: Taylor [1, 2] outlines the motivation for using a modern SIDF approach for control system design and establishes a systematic plan of attack, and both [2] and Taylor and Strobel [3] compare SIDF models with those based on the random-input describing function technique. The SIDF approach was first applied in [3], in which a linear PID compensator in series with a single static nonlinearity is designed, and extended by Taylor and Strobel [4] to the design of a fully nonlinear PID compensator, i.e., a proportional-integral-derivative controller with a nonlinearity in each of the three channels. In each case, it was shown that the nonlinear compensator is capable of reducing amplitude sensitivity or even correcting instabilities caused by the amplitude dependence of the nonlinear plant without unnecessarily sacrificing performance.

This technique uses a set of SIDF models of the nonlinear plant as the basis for nonlinear compensator synthesis. SIDF models are used because they provide an excellent characterization of the major nonlinear effect with which we are concerned: the sensitivity of the nonlinear plant's input/output (I/O) behavior to the amplitude of the input signal; this issue has been discussed in detail in [1, 2, 3]. In summary, given an input in the form $u(t) = u_0 + a_i \cos(\omega t)$ the I/O model is of the form

$$y(t) = y_0 + Re[G(j\omega; u_0, a_i) a_i e^{j\omega t}]$$

where higher harmonics are neglected in this representation. A set of SIDF models corresponding to $\{a_i\}$ is denoted $\{G(j\omega; u_0, a_i)\} = \{G_i\}$.

Once a set of SIDF models is available, the synthesis of a nonlinear compensator proceeds as follows: first, a *linear compensator set* is designed based on these models, with the objective of making the overall open-loop control system as insensitive to input amplitude as possible for a set of error signal amplitudes $\{a_i\}$. This yields a parametrized set of compensators $\{C_i(a_i)\}$, where the configuration of each compensator is the same (e.g., PID) but the parameters differ (e.g., $\{K_{P,i}(a_i)\}$ etc.). Final synthesis of the nonlinear control system is then accomplished by SIDF inversion to determine the required compensator nonlinearities.

The particular approach presented here involves the design of a PI compensator in cascade and a tachometer placed in inner-loop feedback about the plant; hereafter this will be referred to as a PI + Tach controller. In general, there is no restriction as to compensator structure except that the linear controller set and final nonlinear controller must be of the same type.

The work described in this paper includes a new nonlinear compensator synthesis approach and its application to a motor + load model with saturation and stiction. The major extension in comparison to earlier research [4] is that the controller obtained has the derivative action in an inner-loop feedback configuration. This controller configuration is more effective than the cascade PID used previously, as it is well known that differentiation should not be included in a precompensator. This extension is not straightforward because of the assumed amplitude sensitivity of the nonlinear plant.

This approach is capable of treating nonlinear plants of a very general type, with no restrictions as to system order, number of nonlinearities, configuration, or nonlinearity type. These results make the use of SIDF-based nonlinear controller design methods substantially more effective. It is also believed that this design approach will provide a framework for further developments in the realm of compensator design for nonlinear systems.

2. NL PI + TACH DESIGN APPROACH

First, it is important to state the premises of the SIDF design approaches that we have been developing:

1. The nonlinear system design problem being addressed is the synthesis of controllers that are effective for plants having frequency-domain I/O models that are *sensitive* to input amplitude (e.g., for plants that behave very differently for "small" and "large" input signals).
2. The primary objective of nonlinear compensator design is to arrive at a closed-loop system that is as *insensitive* to input amplitude as possible.

This encompasses a limited but important set of problems, for which gain-scheduled compensators cannot be used and for which other approaches (e.g., variable structure systems, model-reference adaptive control, global linearization) do not apply because their objectives are different (e.g., their objectives deal with *asymptotic* solution properties rather than *transient* behavior, or they deal with the behavior of *transformed* variables rather than physical variables).

An outline of the design algorithm for the nonlinear PI + Tach controller is as follows:

1. Select a set of input amplitudes and frequencies that characterizes the behavior of the plant in the operating regimes of interest.
2. Generate SIDF models of the plant corresponding to the input amplitudes and frequencies of interest.
3. Examine SIDF models to qualitatively determine:
   - appropriateness of the design approach,
   - severity of the nonlinear plant amplitude sensitivity, and
   - type of nonlinear controller likely to be needed.
4. Design a nonlinear inner-loop tach feedback controller using a modified D'Azzo and Houpis algorithm [5].
5. Find SIDF models for the nonlinear plant plus nonlinear rate feedback.
6. Design a cascade nonlinear PI compensator using an extension of the frequency-response mapping technique described in [4].
7. Validate the design through simulation.

The resulting compensator structure is shown in Figure 1.

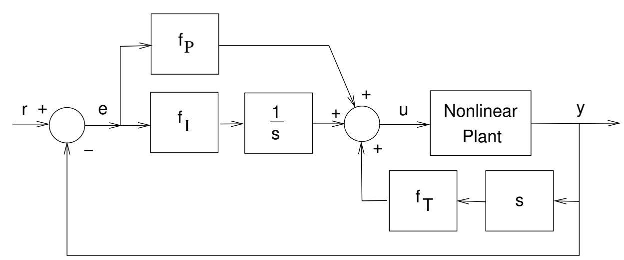


Figure 1: Nonlinear PI + Tach Control Structure

Only items 4 and 6 above will be discussed in detail. The selection of the input amplitudes and frequencies for SIDF generation (item 1) and the simulation to validate the design (item 7) depend on the familiarity of the designer with the system in question. The actual generation of SIDF models (items 2 and 5) is discussed briefly below; see [6] for more details. Finally, the qualitative analysis of SIDF models (item 3) is a current area of research.

2.1 SIDF Generation

The generation of sinusoidal-input describing function models that provide an amplitude-dependent I/O characterization for a nonlinear plant has been dealt with in detail in [3, 6]. There are two basic approaches: solving the nonlinear algebraic equations derived from the principle of harmonic balance, and simulation coupled with Fourier analysis.

The first method is not easy to apply, especially if it is desired to develop a general package that substitutes the appropriate SIDFs into the nonlinear algebraic equations and solves them. Also, the assumption is made that the input to each nonlinearity is approximately sinusoidal (refer to Atherton [7]), which may leave the analysis open to question. However, there is an advantage to this approach: the SIDF model is obtained in a form that lends itself to further analysis such as predicting the existence of limit cycles.

The second technique is easier to implement, given a good package for integrating nonlinear differential equations, and avoids the need to justify the assumption that the inputs of every nonlinearity are nearly sinusoidal—there is no such assumption made using simulation. The only assumption is that a frequency-domain amplitude-dependent I/O model provides a good representation of the behavior of a nonlinear plant for control system design; that issue has been discussed in [2, 3]. In our opinion, while SIDF models are not exact, a *set* of SIDF models covering the range of input amplitudes that will be encountered provides an excellent basis for "robust design", in the sense that the sensitivity of the plant behavior to input amplitude is one of the most important issues in robustness, and the SIDF I/O model is the least conservative model that accurately takes this factor into account.

Extensions have been made to the nonlinear simulation package SIMNON to perform SIDF I/O model generation for nonlinear system models. The basic idea is to drive the nonlinear plant with a sinusoid of the desired amplitude for a number of frequencies of interest, and evaluate Fourier integrals as the simulation proceeds. The simulation for a given frequency is sampled after each cycle and stopped when the Fourier integrals have converged; then the I/O model is evaluated, as described in Taylor [6]. In general this gives us $G(j\omega; u_0, a_i)$ as mentioned above; in applications where a constant offset u_0 is not considered we will use the shorthand notation $G(j\omega; a_i)$ to designate the SIDF I/O model generated by driving a nonlinear system with the input $u(t) = a_i \cos(\omega t)$.

2.2 Inner-Loop Controller Design

The general objective when designing the inner-loop rate feedback controller is to give the same benefits expected in the linear case, namely stabilizing and damping the system, if necessary, and reducing the sensitivity of the system to disturbances and plant nonlinearities (see Thaler [8]). At the same time, we wish to design a nonlinearity to be used with the controller that will desensitize the inner-loop as much as possible to different input amplitudes.

As shown in D'Azzo and Houpis [5], it will be convenient to work with inverse Nyquist plots of the plant I/O model, i.e., inverting the SIDF frequency-response information in complex-gain form and plotting the result in the complex plane. In the linear case, this allows us to study the closed-inner-loop (CIL) frequency response $G_{CIL}(j\omega)$ in the inverse form

$$\frac{1}{G_{CIL}(j\omega)} = \frac{1 + G(j\omega)H(j\omega)}{G(j\omega)} = \frac{1}{G(j\omega)} + H(j\omega)$$

where the effect of $H(j\omega)$ on $1/G_{CIL}(j\omega)$ is easily determined.

The inner-loop tach feedback design algorithm given by D'Azzo and Houpis and referred to as *Case 2* uses a construction amenable to extension to nonlinear systems. For linear systems, this algorithm is based on adding a tachometer and external gain in order to adjust the inverse Nyquist plot to be tangent to a given M-Circle at selected frequency. Referring to the diagram taken from [5] and shown in Figure 2, the al-

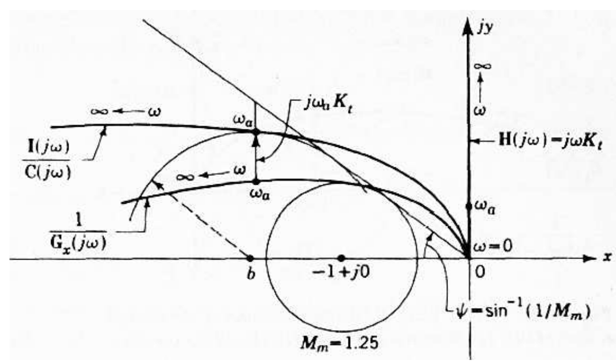


Figure 2: Tach Feedback Design Algorithm 2

gorithm is applied as follows: The selected value of ω_a is first found; the projection of this point on the real axis, point b, will be the center of the scaled M-Circle. The radius of this scaled M-Circle will be chosen to make it tangent to the line defined by the angle ψ. Next K_t is determined to move the $1/G_x(j\omega)$ plot to be tangent to this scaled M-Circle, giving the plot $1/G_{CIL}(j\omega) = I(j\omega)/C(j\omega) \approx 1/G_x(j\omega) + j\omega K_t$. The gain A_2 can then be determined by the distance to point b. Due to the complicated geometry involved, some trial and error may be required.

The algorithm is extended to the nonlinear case by applying it to each SIDF frequency-response model G_i corresponding to each plant input amplitude a_i. Then for each input amplitude a tachometer gain, $K_{t,i}$, and external (to the inner loop) gain $A_{2,i}$ is found. At this point in the design, the $A_{2,i}$ values are not used, since the external gain will be subsumed in the cascade portion of the controller that is synthesized in the next step.

The set of desired tachometer gains $K_{t,i}(a_i)$ is then used to synthesize the tachometer nonlinearity (f_T in Figure 1). As first described in [2], this gain/amplitude data is interpreted as SIDF information for an unknown static nonlinearity. A least-squares routine is used to adjust the parameters of a general piecewise-linear nonlinearity so that the SIDF of that nonlinearity fits this gain/amplitude data with minimum mean square error; this generates the desired controller nonlinearity. While the concept is identical to [3, 4, 6], the nonlinearity used in this study is more general, and the algorithmic implementation allows much more flexibility and interaction in achieving a good fit.

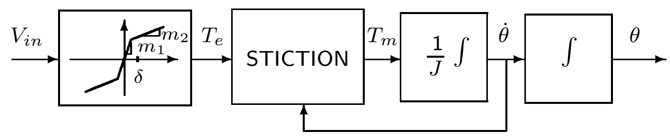
2.3 Cascade PI Controller Design

Referring back to Figure 1 and to the design strategy outlined above, the final step in the complete controller design is generating the nonlinear cascade PI compensator. The general idea is to first generate SIDFs for the nonlinear plant (which, in this approach, is actually the nonlinear plant with nonlinear rate feedback) over the range of input amplitudes and frequencies of interest. This information forms a frequency-response map as a function of both input amplitude and frequency. A single nominal input amplitude is selected, a^*, and a linear compensator is found that best compensates the plant *at that amplitude*. This compensator, in series with the nonlinear plant, is used to calculate the corresponding desired open-loop I/O model $CG^*(j\omega; a)$, the *frequency-domain objective function*. Then, at each input amplitude a_i a least-squares algorithm is used to adjust the parameters of the linear PI compensator, $K_{P,i}(a_i)$ and $K_{I,i}(a_i)$, to minimize the difference between the resulting frequency response found using the linear compensator and interpolating on the SIDF frequency-response map, and $CG^*(j\omega; a)$, as described in [4]. The nonlinear PI compensator is then obtained by synthesizing the nonlinearities f_P and f_I in Figure 1 by SIDF inversion. This algorithm for PI design has also been improved and extended for more general use.

The original algorithm [4] had no mechanism for adjusting the compensator parameter fit to emphasize or deemphasize given frequency ranges; because it used a PID controller, which has three channels covering low, medium, and high frequencies, this ability was not necessary. However, because we now wish to use a PI cascade compensator, and in general don't wish to assume that the compensator will be able to affect all frequency ranges equally, the ability to weight the different frequencies is needed. Additionally, because different input amplitudes correspond to different parts of the system response (e.g., high input amplitudes correspond to high errors which require an emphasis on the transient response while low input amplitudes correspond to low errors and an emphasis on the steady-state response), it is necessary to provide different frequency weights at each input amplitude. This ability has been added, and thus allows frequency-domain ranges to be weighted to achieve desired closed-loop time-domain objectives in a very flexible framework.

3. NL PI + TACH DESIGN EXAMPLE

A controller will be designed for the motor + load model shown in Figure 3, which uses a gain reduction to model



$$T_m = \begin{cases} T_e - f_v\dot{\theta} - f_c\mathrm{sign}(\dot{\theta}), & |T_e| > f_c \text{ or } \dot{\theta} \neq 0.0 \\ 0.0, & \text{otherwise} \end{cases}$$

Figure 3: Motor Model with Saturation and Stiction

motor saturation and also has a stiction nonlinearity. Following the general algorithm presented above, the first step is to perform a SIDF analysis of the motor. This was done using the extension to SIMNON mentioned above, and the resulting I/O model is portrayed in Bode magnitude and phase response plots for the different input amplitudes considered (Figure 5).

As seen from the plots, the magnitude differs by over 14 dB and the phase response has significantly different shapes. Additionally, an examination of the higher harmonics of the SIDF response show them to be several orders of magnitude lower than the fundamental. These two facts indicate the the SIDF approach is both valid and appropriate for this problem. The observation that the phase responses are very different in shape over the frequency range of interest indicates that one static compensator nonlinearity cannot desensitize the open-loop frequency response very well, so we choose to design a fully nonlinear PI + Tach controller.

The next step is the design of the inner-loop feedback of the nonlinear tachometer. The *Case 2* D'Azzo and Houpis algorithm is used, where the desired M-Circle radius and natural frequency of the closed-loop system are selected to be the default value of $\sqrt{2}$ (which should result in acceptable overshoot) and $\omega_a = 25$. This target natural frequency was estimated from the desired response time of the closed-loop system. A routine then implements the geometry relations discussed earlier to develop the gain/amplitude relationship needed for the tach nonlinearity at each plant input amplitude. The gain/amplitude relationship found is shown in Table 1. Note that this design algorithm calls for slight *positive*

Tachometer		PI Controller		
e	K_t	e	K_p	K_i
0.4150	-0.3413	0.050	6.1266	34.0292
1.8091	-0.0385	0.065	5.2276	33.5021
3.5008	-0.0088	0.080	4.7781	32.8287
8.7778	0.0108	0.160	3.7593	30.0063
12.5293	0.0263	0.320	3.5064	28.8968
19.0920	0.0461	0.640	4.5711	22.5852
31.6697	0.0675	1.280	5.8645	8.0451
56.3283	0.0867	2.560	6.4135	0.0000

Table 1: Gain / Amplitude Relationships

feedback at low input amplitudes—this is due to the algorithm's objective which is to counteract the high effective damping provided by the stiction nonlinearity. The final step in the feedback controller design is taking the gain/amplitude information and generating the tachometer nonlinearity. SIDF inversion resulted in a one-segment linear gain with a negative discontinuity at the origin. Its gain/amplitude relationship and fit with the data is shown in Figure 4A.

In this approach, the cascade PI design was done on the basis of SIDF I/O models of the plant with nonlinear tach feedback, since this represents the "plant" for the cascade design. A nominal linear PI controller was designed at a nominal, mid-range, input amplitude, and the SIDF for the linear PI in cascade with the closed-inner-loop plant at that amplitude was generated and denoted $CG^*_{CIL}(j\omega; a^*)$. Using this nominal linear controller as a starting point, the parameters were fitted at each input amplitude to match $CG^*_{CIL}(j\omega; a^*)$ as closely as possible. The gain/amplitude relationships for $K_{P,i}(a_i)$ and $K_{I,i}(a_i)$ are also shown in Table 1. This information was then used to synthesize proportional and integral channel nonlinearities as shown in Figures 4B and 4C.

The final stage of the design was simulation to validate the resulting closed-loop system. The resulting time histories, with step input amplitudes ranging from $r_1 = 0.20$ to $r_8 = 10.2$, are shown in Figure 6A. As compared with the time histories for the nominal linear controller and nonlinear PID controller used in [4], Figures 6B and 6C, respectively, it is evident that the resulting design achieves better performance, both in the sense of lower overshoot and less settling time and in the sense of very low sensitivity of the response over the range of input amplitudes considered.

4. SUMMARY AND CONCLUSIONS

The method outlined in Section 2 is a specific realization of the basic concept of using SIDF I/O models as the basis for nonlinear compensator design proposed in [1, 2]. Based on the example shown in Section 3, and on work in progress, we feel that this approach shows considerable promise in dealing with one of the more difficult problems in nonlinear systems design—the design of controllers to correct for the amplitude-dependence of nonlinear plants.

Note: This is not the original manuscript, which does not exist in electronic form – it was reconstituted in 2004. The simulation results are not exactly the same, as MATLAB was used instead of Simnon.

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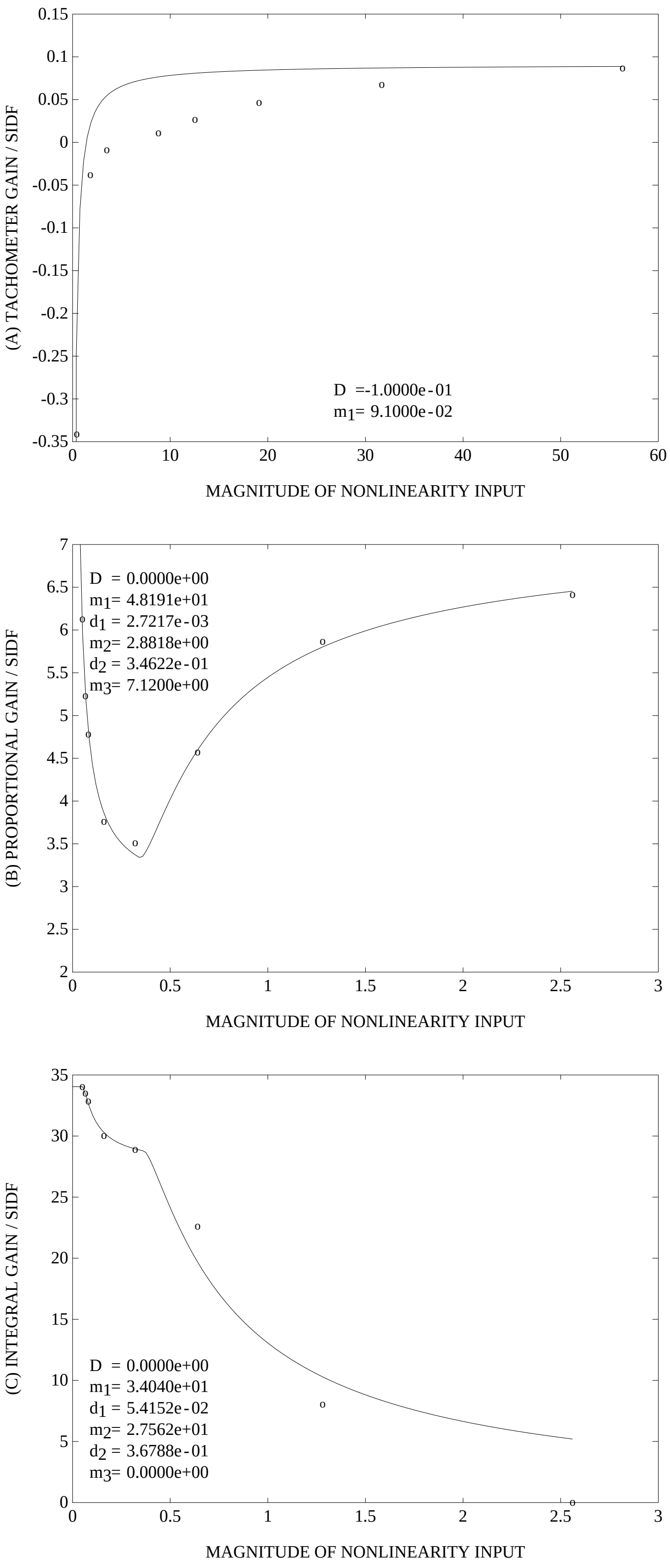

Figure 4: Nonlinearity Gain vs Input Amplitude

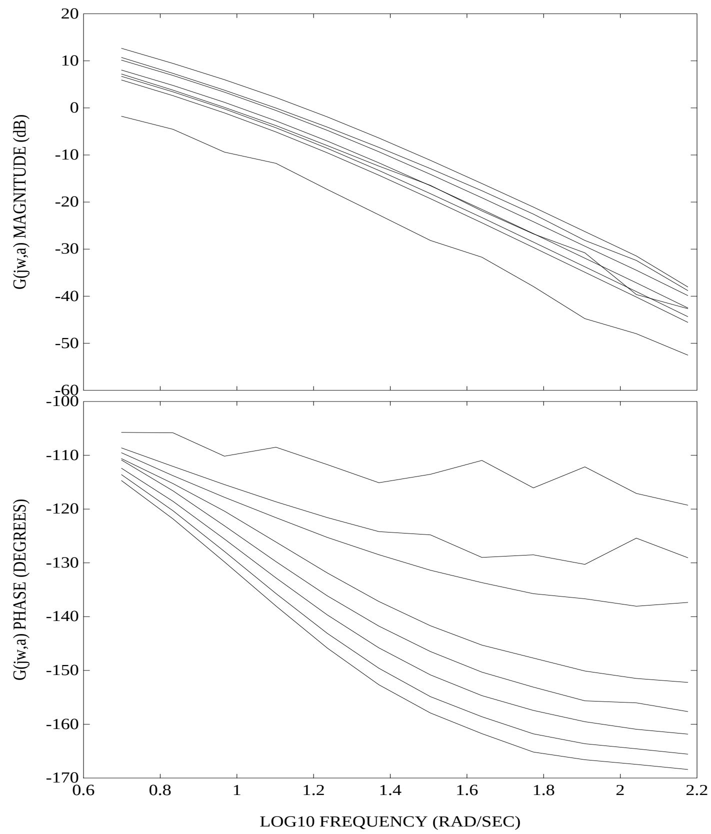

Figure 5: Motor Model $G(j\omega, a)$ Plots

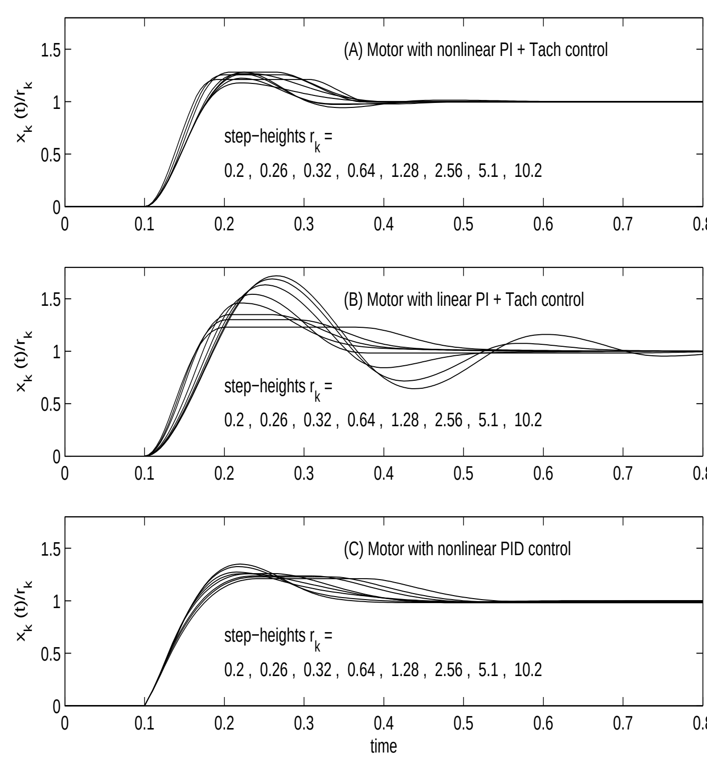

Figure 6: Compensated Motor Step Responses