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MINLP Formulation of Optimal Reactive Power Flow

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Abstract: A method for the solution of optimal reactive power dispatch which treats var sources and transformer tap ratios as discrete variables is presented. The optimal reactive power flow (ORPF) problem is inheritly a mixed-integer nonlinear programming (MINLP) problem. Due to difficulty of solution, this problem is often approximated as a nonlinear programming (NLP) problem. The NLP methods find a suboptimal solution in most cases. For finding the global optimal solution of this problem, an MINLP formulation is proposed and executed. In this formulation, discrete variables, var sources and tap ratios, are modeled as binary variables.

The MINLP problem with only continuous and binary variables is solved by an outer-approximation/equality-relaxation (OA/ER) algorithm. In this algorithm, the MINLP problem is decomposed into a mixed-integer linear programming (MILP) master problem, and an NLP subproblem. These two subproblems are solved successively until convergence criteria are met. A sample network is used for testing the proposed method. The results verify that the MINLP approach can find the global optimum, while NLP algorithms give a suboptimal solution.

Keyword: optimal reactive power flow, power loss minimization, MINLP problems, OA/ER algorithm

1 Introduction

The optimal reactive power dispatch problem has been formulated since the 1960's [1]. Many formulations have been developed since then [2]. This problem is formulated as an NLP problem in most cases. The ORPF is a static mixed-integer nonlinear optimization problem. The main objectives of ORPF study address two important aspects in power systems. The first objective is to maintain the voltage profile of the network in an acceptable range. The second objective is to minimize the total power loss of the network while satisfying the first objective [3, 4].

The control variables for this study include vars/voltages of generators, the tap ratios of transformers and reactive power generation of var sources. The constraints include the var/voltage limits of generators, the voltage limits of load buses, tap ratio limits, var source limits, power flow balance at all buses, and security constraints.

This problem has been solved by linear programming, parametric linear programming, successive linear programming, quadratic programming, gradient method, and nonlinear quadratic programming [2]. In all of these methods, the discrete variables, var sources and tap ratios, are treated as continuous variables in the optimization process. These variables are rounded off to the nearest discrete values after finding the optimal solution. In general, these approaches yield a suboptimal solution. For finding the global optimal solution, the ORPF problem is formulated as an MINLP problem in this paper. In this formulation, both var sources (capacitor and/or reactor banks) and tap ratios are treated as discrete variables. The discrete variables are modeled in terms of binary variables.

Different methods for solving MINLP problems are proposed in the literature. Among these methods, generalized bender decomposition (GBD) [5], branch and bound [6], and OA/ER [7] can be cited. The method which is used in this paper is an *Outer-Approximation/Equality Relaxation* (OA/ER) algorithm [7], which is mainly based on the linearity of binary variables and the convexity of nonlinear functions.

The rest of this paper is organized as follows: in section 2 the general formulation of an MINLP problem is presented. In section 3 the OA/ER algorithm as a solution method for MINLP problems is explained. In section 4 the application of the OA/ER algorithm to ORPF problem is addressed. In section 5 the proposed method is tested on a sample network. We conclude with a short summary of our results and their significance.

2 Formulation of an MINLP Problem

In this section the formulation of a particular class of MINLP problems is addressed. In the class of MINLP which is formulated here, the integer variables are binary numbers and can only have a linear structure, i.e., they don't appear in any nonlinear terms in objective function or constraints. It is also assumed that the nonlinear functions are convex and differentiable at least up to the second order.

In the following formulation, the real and binary variables are denoted by x and y vectors, respectively. The MINLP formulation with the mentioned assumptions can be written as:

$$\begin{aligned} &\min\ Z = f(x) + c^T y \\ &\text{subject to:} \\ &h(x) + Ay = 0, \\ &g(x) + By \leq 0, \\ &x_{\min} \leq x \leq x_{\max}, \end{aligned} \tag{1}$$

where $f\colon \Re^n \to \Re$, $h\colon \Re^n \to \Re^r$, $g\colon \Re^n \to \Re^p$, $x \in \Re^n, y \in [0,1]$; A, B are matrices with compatible dimensions; and c is a column vector.

3 Solution Method

The MINLP problem as formulated in (1) can be solved by several different techniques. Some of the well known methods are the Generalized Bender Decomposition (GBD), the OA/ER algorithm, and a branch and bound method with NLP subproblems. The Branch and bound technique is appropriate for problems in which the solution of the relaxed NLP is not expensive, or the number of NLP subproblems in the search tree is very limited. GBD approach in comparison with OA/ER algorithm needs more major iterations where NLP subproblems and MILP master problems are solved successively, however, its MILP problem has a smaller size than OA/ER [8]. The method which is used in this paper is an (OA/ER) algorithm [7], which is chosen mainly based on the linearity of binary variables and the convexity of nonlinear functions.

In the OA/ER algorithm the continuous and discrete optimizations are decomposed. Continuous optimization is performed in NLP subproblems. The discrete optimization is performed in a MILP master problem. The algorithm involves successive solutions of MILP and NLP subproblems, until stopping criteria are met [7]. In the following sections the formulation of these two subproblems are explained.

3.1 Formulation of NLP Subproblem

The optimization of an MINLP problem with only continuous variables is performed in this section. In the NLP subproblem, the binary variables of MINLP problem are fixed at their optimal values found in the last MILP solution. By this process, the MINLP problem will be transformed to the following NLP subproblem:

$$\begin{aligned} &\min\ Z^k_{\text{NLP}} = f(x) + c^T y^k \\ &\text{subject to:} \\ &h(x) + Ay^k = 0, \\ &g(x) + By^k \leq 0, \\ &x_{\min} \leq x \leq x_{\max}, \end{aligned} \tag{2}$$

where y^k are constant values found in the previous MILP.

3.2 Formulation of Master Problem

In the master problem a mixed-integer linear programming problem is optimized. This formulation can be written as [8]:

$$\begin{aligned} &\min\ Z^k_{\text{MILP}} = \alpha + c^T y \\ &\text{subject to:} \\ &f(x^k) + \nabla f(x^k)^T (x - x^k) - \alpha \leq 0, \\ &T^k [h(x^k) + \nabla h(x^k)^T (x - x^k) + Ay] \leq 0, \\ &g(x^k) + \nabla g(x^k)^T (x - x^k) + By \leq 0, \\ &{Z^*_{\text{MILP}}}^{k-1} \leq c^T y + \alpha, \\ &x_{\min} \leq x \leq x_{\max}, \end{aligned} \tag{3}$$

where α is an upper bound for $f(x)$; $\nabla f(x^k)$ is the n-column gradient vector of $f(x)$ at x^k; $\nabla h(x^k)$, $\nabla g(x^k)$ are the $n \times r$ and $n \times p$ Jacobian matrices of $h(x)$ and $g(x)$ at x^k, respectively; ${Z^*_{\text{MILP}}}^{k-1}$ is the optimal value of objective function of the previous MILP solution; T^k is an $r \times r$ diagonal matrix with diagonal terms as:

$$t^k_{ii} = \begin{cases} -1 & if \quad \lambda^k_i < 0 \\ +1 & if \quad \lambda^k_i > 0 \\ \ \ 0 & if \quad \lambda^k_i = 0 \end{cases} \quad i = 1, 2, \cdots, r; \tag{4}$$

where λ^k_i are the optimal Lagrangian multipliers for the equality constraints $h_i(x) = 0$ $(i = 1, \cdots, r)$ found in the last NLP subproblem. The formulation given in (3) is a relaxed MILP problem.

3.3 OA/ER Algorithm

In performing the OA/ER algorithm the formulations given for NLP subproblem (2), and MILP master problem (3), will be used. Different steps of this algorithm can be summarized as follows:

Step 1: solve the relaxed MINLP problem given in (1) by replacing the binary variables, y, by real variables limited between 0 and 1 by an NLP algorithm. The solution can be feasible or infeasible. If the solution is infeasible, either the original MINLP problem is infeasible, or the initial point is not good. If the solution is feasible, and y^0 is integer the algorithm stops. Otherwise, y^0 is rounded to the nearest discrete values. Then, set $k = 0$, ${Z^*_{\text{MILP}}}^{k-1} =$

$-\infty$, $x^* = x^0$, $y^* = y^0$, Z^*_{NLP} equal to the optimal value of objective function, and go to step 2.

Step 2: solve the following MILP master problem with (x^k, y^k) as initial points:

$$
\begin{aligned}
&\min\ Z^k_{\text{MILP}} = \alpha + c^T y, \\
&\text{subject to:} \\
&\left.\begin{aligned}
&f(x^i) + \nabla f(x^i)^T (x - x^i) - \alpha \le 0, \\
&T^i[h(x^i) + \nabla h(x^i)^T (x - x^i) + Ay] \le 0, \\
&g(x^i) + \nabla g(x^i)^T (x - x^i) + By \le 0,
\end{aligned}\right\} i = 0, 1, , \cdots, k \\
&\sum_{j \in B^i} y_j - \sum_{j \in N^i} y_j \le |B^i| - 1, \quad i = 0, 1, \cdots, k, \textit{(integer cuts)} \\
&Z^{*\,k-1}_{\text{MILP}} \le c^T y + \alpha, \\
&x_{\min} \le x \le x_{\max},
\end{aligned}
\tag{5}
$$

where sets $B^i = \{j: y_j = 1\}$, $N^i = \{j: y_j = 0\}$; and $|B^i|$ is the cardinality of B^i. In (5), integer cuts have been produced to eliminate the previously-determined integer vectors $y^0, y^1, y^2, \cdots, y^k$ from further consideration. Two cases are possible:

(a) If the optimal integer solution y^{k+1} exists with objective value $Z^{*\,k}_{\text{MILP}} \le Z^*_{\text{NLP}}$ go to step 3.

(b) If $Z^{*\,k}_{\text{MILP}} \ge Z^*_{\text{NLP}}$ or no feasible solution exists, stop. The optimal solution is Z^*_{NLP} at x^*, y^*.

Step 3: solve the NLP subproblem for fixed $y = y^{k+1}$ to find optimal point (x^{k+1}, y^{k+1}) and $Z^{*\,k+1}_{\text{NLP}}$.

Step 4: (a) if the NLP subproblem is feasible and $Z^{*\,k+1}_{\text{NLP}} < Z^*_{\text{NLP}}$, set $Z^*_{\text{NLP}} = Z^{*\,k+1}_{\text{NLP}}$, $x^* = x^{k+1}$, $y^* = y^{k+1}$, $k \leftarrow k+1$, and go back to step 2 by adding the integer cut.

(b) If the NLP subproblem is feasible and $Z^{*\,k+1}_{\text{NLP}} > Z^*_{\text{NLP}}$ or the NLP subproblem is infeasible, set $k \leftarrow k+1$ and go back to step 2 by adding the integer cut to eliminate y^{k+1} from the solution space.

4 Application of OA/ER to ORPF problem

In this section, the formulation of the optimal reactive power flow problem is addressed. This formulation is rearranged as an MINLP problem in two steps. In the first part capacitor and/or reactor banks (var sources) are formulated as discrete variables. In the second section, transformers tap ratios are modeled as discrete variables.

4.1 Formulation of ORPF problem as an NLP problem

The formulation of ORPF problem with continuous var sources and tap ratios can be written as:

$$
\begin{aligned}
&\min\ P_L, \\
\text{subject to:}\quad &h(x) = 0, \\
&g(x) \le 0, \\
&x_{\min} \le x \le x_{\max},
\end{aligned}
\tag{6}
$$

where P_L is the total power loss of transmission lines.

4.2 Formulation of ORPF problem as an MINLP problem

In this formulation both var sources and transformer tap ratios can be treated as discrete variables. The discrete variables are modeled in terms of binary variables.

4.2.1 Formulation of Discrete Var Sources

For a capacitor bank with N_{Ci} equal units, one possible formulation of var value in terms of binary variables can be written as:

$$
Q_{Ci} = \Delta Q_{Ci} \sum_{j=1}^{j_{\max}} 2^{(j-1)}\, y^j_{Ci}, \tag{7}
$$

where Q_{Ci} is the var value of capacitor bank i, ΔQ_{Ci} is the var value of each unit, $j_{\max} = \log_2 N_{Ci}$, and $y^j_{Ci} \in [0, 1]$ are the related binary variables. By adding (7) as new constraints to the NLP formulation given in (6), the MINLP formulation with discrete var sources will be obtained. The formulation for reactor banks is similar.

4.2.2 Formulation of Discrete Transformer Tap ratios

In the following formulation, it is assumed that all the tap steps have equal values of ΔT_i, and the number of steps is N_{Ti}. The discrete tap ratios in terms of binary variables can be formulated as:

$$
\begin{aligned}
&T_i = T_{i\min} + \Delta T_i \sum_{j=1}^{j_{\max}} 2^{(j-1)}\ y^j_{Ti}, \\
&T_{i\min} \le T_i \le T_{i\max},
\end{aligned}
\tag{8}
$$

where T_i is the tap ratio of transformer i, $j_{\max} = \log_2 N_{Ti}$, and y^j_{Ti} $(j = 1, 2, \cdots)$ are the related binary variables. T_i can appear in any nonlinear terms, while the related binary variables appear only in linear relations of (8). By adding (8) as a new set of constraints to the MINLP problem found in section 4.2.1, the MINLP formulation with discrete var sources and transformers tap ratios will be defined.

5 Simulation Results

The 6-bus Ward-Hale system (Fig. 1) is used for the test of the OA/ER algorithm. The line data and bus data of the 6-bus system are given in Tables 1 and 2, respectively. The data in Table 2 corresponds to full load conditions. The limits of bus voltages, tap ratios, shunt capacitors, and generator vars are given in Table 3. In the simulation of the sample network, the capacitors and transformer taps are treated as discrete variables. For comparison purposes, the results of conventional ORPF (which uses the NLP algorithm) are also presented.

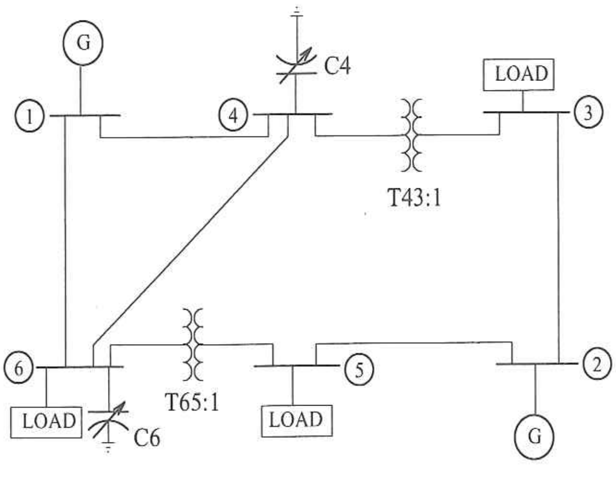

Figure 1: Ward-Hale 6-bus System

5.1 ORPF problem with Continuous Variables

The ORPF problem may be solved by an NLP algorithm, if the discrete variables are assumed to be continuous during the optimization process. Their values are rounded off to the nearest discrete values in the last iteration. For this study, the loading conditions given in Table 2 are used. The computed optimal values of control variables from continuous solution are given in the second column of Table 4. The discrete variables are rounded off to the nearest integer values with the following two methods:

1. rounding all the discrete variables in one step. From the continuous solution, the optimal value of Q_{C6} is at its limit, and there is no need for rounding its value. The nearest integer value of Q_{C4} is zero. These results with rounded tap ratios are given in the third column of Table 4.

2. If the step size of capacitors is much bigger than the transformer taps, as in this case, it is advisable to round off the discrete variables in two steps. In the first step, the capacitors are rounded off to the nearest integer values. Afterwards the ORPF program is run with the new fixed values of capacitors. Then the new optimal values of tap ratios are rounded off to the nearest discrete values. The results are given in the fourth column of Table 4.

In both cases, it is necessary to run the ORPF program after the final round offs to optimize the continuous control variables.

Table 1: Line data of the 6-bus system on 100 MVA base

Line Number	Bus Number		Impedance (per unit)		Tap Ratio
	From	To	R	X	
1	1	6	0.123	0.518	–
2	1	4	0.080	0.370	–
3	4	6	0.097	0.407	–
4	6	5	0.000	0.300	1.025
5	5	2	0.282	0.640	–
6	2	3	0.723	1.050	–
7	4	3	0.000	0.133	1.100

Table 2: Bus data of the 6-bus system in full load conditions

Bus Number	Voltage		Load	
	V (per unit)	δ (deg)	P (MW)	Q (MVAR)
1	1.04	0.0	0.0	0.0
2	1.11	-6.6	0.0	0.0
3	0.85	-14.0	55.0	11.0
4	0.95	-10.1	0.0	0.0
5	0.92	-13.6	30.0	18.0
6	0.91	-12.8	50.0	10.0

Table 3: Low and high limits of variables

Dependent Variables	Limits		Control Variables	Limits	
	Low	High		Low	High
V_{g1}	1.00	1.10	Q_{g1}	-20.0	100.0
V_{g2}	1.10	1.15	Q_{g2}	-20.0	100.0
V_{l3}	0.90	1.00	Q_{C4}	0.0	15.0
V_{l4}	0.90	1.00	Q_{C6}	0.0	30.0
V_{l5}	0.90	1.00	$T43$	0.95	1.09
V_{l6}	0.90	1.00	$T65$	0.95	1.09

Table 4: Continuous and discrete optimal values of capacitors and taps (ratios)

Variable	continuous capacitors & taps	round off in one step	round off in two steps	MINLP with discrete capacitors & taps
Q_{C4}(Mvar)	9.06	0.0	0.0	20.0
Q_{C6}(Mvar)	25.00	25.0	25.0	25.0
$T43$	0.982	0.98	0.98	0.98
$T65$	1.055	1.06	1.05	1.08
P_L(MW)	8.51	8.68	8.62	8.57

5.2 ORPF problem with Discrete Var Sources and Tap Ratios

The var sources and transformer taps are modeled as discrete variables in an MINLP formulation. The optimal results for this case are given in the fifth column of Table 4. By comparing the results of this section with results of the NLP approach, it is clear that the NLP method gives a suboptimal solution, while the OA/ER algorithm finds the global optimum solution. The total power loss which is found by OA/ER algorithm is 1.3% less than that from the NLP with rounding off in one step, and 0.6% with round off in two steps.

By studying some other simulation results, it was also observed that the discretization of tap ratios has less effect than that from var sources. This result is due to the small step size of tap ratios. If a larger step size for transformers taps or smaller step size for capacitor banks is selected, the results could be different.

6 Discussion and Conclusions

The MINLP formulation, and the basic concept of the OA/ER algorithm as a tool for MINLP solution are presented. The ORPF problem is inheritly an MINLP problem, but due to the complications, this problem is solved by NLP algorithms in most applications. In the NLP approach the discrete variables, var sources and transformers tap ratios, are treated as continuous variables. The optimal values of continuous variables are rounded off to the nearest discrete values at the final iteration of the NLP program. This procedure results in a suboptimal solution. Two methods are considered for rounding the continuous variables. The difference between the simulation results shows that the rounding method used for finding the final solution is of great importance. The two methods are different by 0.7% in the total power loss result.

The ORPF problem as an MINLP problem is formulated with discrete var sources and tap ratios. The sample network is studied for NLP and MINLP cases. The optimal solution found by the OA/ER algorithm is the global minimum of the problem, and has less power loss than that from the NLP approach. The total power loss in MINLP approach has a reduction of 0.6% to 1.3% in comparison to the NLP approach. This demonstrates that the existing NLP approaches give a suboptimal solution for the ORPF problem, while the MINLP approach can find the golobal optimal solution.

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