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DESCRIBING FUNCTION METHODS FOR HIGH-ORDER HIGHLY NONLINEAR SYSTEMS

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ABSTRACT

Applying the modern systems approach to studying the "quality of life" often entails developing and using complicated dynamic models (high-order, highly nonlinear differential equations). Dealing with mathematical models of this sort can be difficult. In some instances, however, these models can be effectively studied using recent advances in describing function theory. These methods are described and illustrated in this paper.

KEYWORDS

Large scale systems; highly nonlinear systems; describing function methods; limit cycle analysis; statistical performance analysis; nonlinear covariance analysis.

INTRODUCTION

The complexity of problems facing modern society is rapidly outstripping man's capability to manage them on the basis of "intuition" and the old "rules of thumb" that have been successful in the past. The systems theoretic answer to this difficulty is to develop analytical models (usually dynamic) that account for the complexity of the system, and study the resulting model behavior via analytical methods or simulation. Analysis and simulation are the usual bases for both validating the model (determining if the model is realistic) and for problem solving (determining how to control the system or modify the system so that it behaves more appropriately).

Many models that result from such activity are differential equations that are of high order (having many state variables) and nonlinear. In some circumstances, the nonlinearity can be neglected, and well-established linear system analysis and design methods can be used for predicting system behavior and for problem solving. Often, however, nonlinearity cannot be neglected and the systems engineer feels forced to rely solely on simulation as a tool for performance evaluation and design.

The last decade has seen substantial progress in extending quasi-linearization or describing function techniques to permit the analytic study of high-order highly nonlinear dynamic systems models. Three areas are especially noteworthy:

- Limit Cycle Analysis - Until recently, nonlinear systems that exhibit limit cycle (oscillatory) behavior could not be analyzed unless they were low-order or contained only one dominant nonlinearity. New extensions of sinusoidal-input describing function (SIDF) theory (Taylor, 1975, 1980; Hannebrink, Lee, Weinstock and Hedrick, 1977) permits systems of any order, with any number of nonlinear effects, to be treated analytically.

- Sinusoidal Input Response Analysis - The same generalized SIDF approach allows the direct determination of the response of complicated nonlinear systems to sinusoidal inputs (Taylor and Mohan, 1980).

- Random Input Response Analysis - Many systems are driven by random inputs (e.g. first-order Markov processes or white noise) or have random parameters. The response or performance of such a system is generally described statistically (e.g. rms vibration levels may provide a measure of ride quality in a train or bus). The statistical behavior of high-order nonlinear systems can now be determined by an analytic technique (Kazakov, 1965; Gelb and Warren 1973) that combines covariance analysis with random-input describing functions (RIDF's); this method has been proven to be much more effective than monte carlo simulation in many instances.

All of the relevant describing function techniques are outlined in this paper.

QUASILINEARIZATION/DESCRIBING FUNCTION METHODS

DF Definitions

The basic idea of the describing function (DF) approach for studying nonlinear system behavior is to replace each system nonlinearity with a linear term whose "gain" is a function of "input amplitude", where the type of input signal is assumed in advance; this concept is dealt with very thoroughly in Gelb and Vander Velde (1968) and Atherton (1975). In this paper, two cases are considered:

Sinusoidal-Input Describing Functions (SIDF's)

$$f(b + a \sin \omega t) \cong f_o(a,b) + n_s(a,b) \cdot a \sin \omega t \qquad (1)$$

Random-Input Describing Functions (RIDF's)

$$f(m + r) \cong \hat{f}(m,\sigma) + n_r(m,\sigma) \cdot r, \qquad r \sim N(0,\sigma) \qquad (2)$$

[m is the mean value of the input, r is a zero-mean gaussian random variable with standard deviation or rms value σ.]

The DF elements (f_o, n_s or $\hat{f}$, n_r) are mathematically formulated to minimize the approximation error in (1) and (2). The usefulness of DF methods lies in the subsequent treatment of the resulting quasi-linear model using linear system analytic techniques, which are well established and usually very straightforward to apply. The power of DF methods is derived from the amplitude-dependence of the DF elements, which accounts for one of the basic effects of nonlinearity. Standard linearization (small-signal or Taylor series linearization) fails to capture this essential property of nonlinear phenomena. A wide variety of DF's, both sinusoidal- and random-input, are catalogued in Gelb and VanderVelde (1968) and Atherton (1975), so we will not consider that aspect of DF theory further.

Limit Cycle Analysis

To illustrate the fundamentals of limit cycle analysis, consider the following autonomous nonlinear differential equation:

$$\dddot{x} + 2\ddot{x} + [2\dot{x} + \mathrm{sgn}(\dot{x})] + [3x + x^3] = 0 \tag{3}$$

The two nonlinearities are quasilinearized as follows: Assume that $x \cong a \sin \omega t$, implying that $\dot{x} \cong a\omega \cos \omega t$. Then (Atherton, 1975)

$$x^3 \cong \frac{3}{4} a^2 \cdot a \sin \omega t$$

$$\mathrm{sgn}\ (\dot{x}) \cong \frac{4}{\pi\ a\ \omega} \cdot\ a\omega \cos \omega t$$

The quasilinearized system then has the characteristic equation

$$s^3 + 2s^2 + \alpha\ s + \beta = 0, \quad \begin{cases} \alpha = 2 + \dfrac{4}{\pi\ a\ \omega} \\ \beta = 3 + \dfrac{3}{4} a^2 \end{cases} \tag{4}$$

Two roots of this equation are in the imaginary axis -- as demanded by the limit cycle assumption -- if

$$2\alpha = \beta \tag{5}$$

and the corresponding natural frequency is

$$\omega = \sqrt{\beta/2} \tag{6}$$

Combining (4) to (6) gives the limit cycle condition to be

$$2 \left[\frac{3}{2} (1 + \frac{a^2}{4}) \right]^{1/2} + \frac{4}{\pi a} = \left[\frac{3}{2} (1 + \frac{a^2}{4}) \right]^{3/2} \tag{7}$$

If a value or values of a can be found that satisfies (7), then SIDF theory predicts that a limit cycle of amplitude a and frequency ω given by (6) exists; the single solution is $a = 1.63$, $\omega = 1.58$.

This example illustrates the following most general limit cycle conditions (Taylor, 1975): Given

$$\underline{\dot{x}} = \underline{f}(\underline{x},\underline{u}) \tag{8}$$

when $\underline{x}$ is an n-dimensional state vector and $\underline{u}$ is an m-dimensional input vector. Assuming that $\underline{u}$ is a vector of constants, denoted $\underline{u}_o$, it is desired to determine if (8) may exhibit LC behavior.

As before, we assume that the state variables are nearly sinusoidal,

$$\underline{x} \cong \underline{x}_c + \mathrm{Re}[\underline{a} \exp (j\omega t)] \tag{9}$$

where $\underline{a}$ is a complex amplitude vector and $\underline{x}_c$ is the state vector center value (which is not a singularity, or solution to $\underline{f}(\underline{x}_o, \underline{u}_o) = \underline{0}$ unless the nonlinearities satisfy certain stringent symmetry conditions with respect to $\underline{x}_o$). Then we again neglect higher harmonics, to make the approximation

$$\underline{f}(\underline{x},\underline{u}_o) \cong \underline{f}_{DF}(\underline{x}_c,\underline{a},\underline{u}_o) + \text{Re}[F_{DF}(\underline{x}_c,\underline{a},\underline{u}_o)\ \underline{a}\ \exp(j\omega t)] \tag{10}$$

The real vector $\underline{f}_{DF}$ and the matrix F_{DF} are obtained by taking the fourier expansions of the elements of $\underline{f}(\underline{x}_c + \text{Re}\ \underline{a}\ \exp(j\omega t),\ \underline{u}_o)$ as illustrated above, and provide the quasi-linear or describing function representation of the nonlinear dynamic relation. The assumed limit cycle exists for $\underline{u} = \underline{u}_o$ if $\underline{x}_c$ and $\underline{a}$ can be found so that

$$\begin{aligned} &\text{(i)} \quad \underline{f}_{DF}(\underline{x}_c,\underline{a},\underline{u}_o) = 0 \\ &\text{(ii)} \quad [j\omega I - F_{DF}(\underline{x}_c,\underline{a},\underline{u}_o)]\underline{a} = 0,\ \underline{a} \neq \underline{0} \end{aligned} \tag{11}$$

(F_{DF} has a pair of pure imaginary eigenvalues, and $\underline{a}$ is the corresponding eigenvector.)

The nonlinear algebraic equations (11) are often difficult to solve. An iterative method, based on successive approximation, can be used successfully for more complicated problems such as the highly complex aircraft performance analysis problem (9 state variables) described in Taylor (1980).

Frequency Response Analysis

The system model is the same as (8), except $\underline{u}$ is a vector of sinusoidal inputs, $\underline{u} = \underline{u}_o + \text{Re}(\underline{c}\ e^{j\omega t})$. Then $\underline{x}(t)$ has the form indicated in (9), and we quasilinearize $\underline{f}(\underline{x},\underline{u})$ as follows:

$$\begin{aligned} &\underline{f}(\underline{x}_c + \text{Re}(\underline{a}\ e^{j\omega t}),\ \underline{u}_o + \text{Re}(\underline{c}\ e^{j\omega t})) \\ &\quad \cong \underline{f}_o(\underline{x}_c,\ \underline{a},\ \underline{u}_o,\ \underline{c}) + \text{Re}[F_{DF}(\underline{x}_c,\ \underline{a},\ \underline{u}_o,\ \underline{c})\ \underline{a}\ e^{j\omega t}] \\ &\quad + \text{Re}[G_{DF}(\underline{x}_c,\ \underline{a},\ \underline{u}_o,\ \underline{c})\ \underline{c}\ e^{j\omega t}] \end{aligned} \tag{12}$$

Applying the same conditions of harmonic balance that underlie the limit cycle condition (11), it is possible to solve for $\underline{x}_c$ and $\underline{a}$ using

$$\begin{aligned} &\underline{f}_o(\underline{x}_c,\ \underline{a},\ \underline{u}_o,\ \underline{c}) = \underline{0} \\ &\underline{a} = (j\omega I - F_{DF})^{-1}\ G_{DF}\ \underline{c} \end{aligned} \tag{13}$$

These 2n nonlinear algebraic equations (n of which are complex-valued) can readily be solved using standard computer routines. In this case, one should be careful to ensure that F_{DF} does not have eigenvalues on or very close to the imaginary axis; otherwise limit cycles may exist, in contradiction to the assumption underlying (9). More examples and details of this approach to frequency response analysis are available in Taylor and Mohan (1980).

Statistical Performance Analysis of Stochastic Systems

The dynamics of a nonlinear continuous-time stochastic system can often be represented by a first-order vector differential equation in which $\underline{x}(t)$ is the system state vector and $\underline{w}(t)$ is a forcing function vector,

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x},t) + G(t)\,\underline{w}(t) \tag{14}$$

The state vector is composed of any set of variables sufficient to describe the behavior of the system completely. The forcing function vector $\underline{w}(t)$ represents disturbances as well as control inputs that may act upon the system. In what follows, $\underline{w}(t)$ is assumed to be composed of a mean or deterministic value $\underline{b}(t)$ and a random input $\underline{u}(t)$, the latter being comprised of elements which are uncorrelated in time; that is, $\underline{u}(t)$ is a white noise process having the spectral density matrix $Q(t)$. Similarly, the state vector has a deterministic component $m(t)$ and a random part $\underline{r}(t)$; for simplicity, $\underline{m}(t)$ will usually be called the mean vector. Thus the state vector $\underline{x}(t)$ is described statistically by its mean vector and covariance matrix, $S(t)$.

The differential equations that govern the propagation of the mean vector and covariance matrix for the system described by (14) can be derived directly, as demonstrated in Jazwinski (1970), to be

$$\begin{aligned} \dot{\underline{m}} &= E\,[\underline{f}(\underline{x},t)] + \underline{G}(t)\underline{b} \\ &\triangleq \hat{\underline{f}} + G(t)\underline{b} \\ \dot{S} &= E\,[\underline{f}\,\underline{r}^T] + E\,[\underline{r}\,\underline{f}^T] + G(t)Q\,G^T(t) \triangleq NS + SN^T + GQG^T \end{aligned} \tag{15}$$

The equation for S has been put into a form analogous to the covariance equations corresponding to $\underline{f}$ being linear, by defining the auxiliary matrix N as above.

The quantities $\hat{\underline{f}}$ and N defined in (15) must be determined before one can proceed to solve for $\underline{m}$ and S. Evaluating the indicated expected values requires knowledge of the joint probability density function (joint pdf) of the state variables. While it is possible, in principle, to evolve the n-dimensional joint pdf $p(\underline{x},t)$ for a nonlinear system with random inputs by solving a set of partial differential equations known as the Fokker-Planck equation or the forward equation of Kolmogorov (Jazwinski, 1970), this procedure is generally not feasible from a practical point of view. The fact that the pdf is not available precludes the exact solution of (15).

One procedure for obtaining an approximate solution to (15) is to <u>assume</u> the form of the joint probability density function of the state variables in order to evaluate $\underline{f}$ and N. Although it is possible to use any joint pdf, all work to date has been based on the assumption that the state variables are jointly normal; the choice was made because it is both reasonable and convenient. The justifications of this approximation are given more fully in Taylor, Price, Siegel and Gelb (1980). The outcome is that $\hat{\underline{f}}$ and N are evaluated by using RIDF's, as in (2), and the numerical integration of (15) can be accomplished readily.

The following problem exemplifies the above RIDF/covariance analysis method of stochastic system performance assessment: Given

$$\ddot{x} + 2\dot{x} + x + x^3 = w, \quad w \sim N(0,q)$$

The zero-mean RIDF for x^3 is $3s_{11}$ where s_{11} is the variance of x (Atherton, 1975); defining the state vector to be $\underline{x}^T = [x \quad \dot{x}]$ gives us the following RIDF matrix for use in (15):

$$N = \begin{bmatrix} 0 & 1 \\ -(1 + 3s_{11}) & -2 \end{bmatrix}$$

Equation (15) then expands to yield the following differential equations for the elements of S:

$$\dot{s}_{11} = 2 \; s_{12}$$
$$\dot{s}_{12} = s_{22} - (1 + 3 \; s_{11}) \; s_{11} - 2 \; s_{12}$$
$$\dot{s}_{22} = -2 \; (2 \; s_{22} + (1 + 3 \; s_{11}) \; s_{12}) + q$$

Given suitable initial conditions, these differential equations can be integrated easily on a digital computer. The steady-state value of S can be obtained directly by setting $\dot{S} = 0$:

$$S_{ss} = \begin{bmatrix} \frac{\sqrt{1 + 3q} - 1}{6} & 0 \\ 0 & \frac{q}{4} \end{bmatrix}$$

More examples and detailed discussions of the theory and use of this technique are given in Taylor, Price, Siegel and Gelb (1980).

Summary and Conclusions

Before the foregoing recent advances in DF theory were made, it was generally conceded that direct simulation of highly nonlinear, high-order differential equations was the only available approach for studying their behavior. Unless a very large number of simulations are performed (with many initial conditions and input functions, for simulation time intervals long enough to allow transients to die away, etc.), the information gained by simulation is often spotty and of questionable value. The analysis tools outlined in this paper have proven to be of great value in supplementing and complementing direct simulation for understanding the behavior of nonlinear systems.

References

Atherton, D. P. (1975). Nonlinear Control Engineering. Van Nostrand Reinhold, London.

Gelb, A. and Vander Velde, W. E. (1968). Multiple-Input Describing Functions and Nonlinear System Design. McGraw-Hill, New York.

Gelb, A. and Warren, R. S. (1973). Direct Statistical Analysis of Nonlinear Systems: CADET. AIAA J., 11, 689-694.

Hannebrink, D. N., Lee, H. S. H., Weinstock, H. and Hedrick, J. K. (1977).

Influence of Axle Load, Track Gage, and Wheel Profile on Rail-Vehicle Hunting. Trans. ASME - J. Eng. Ind., Feb., 186-195.

Jazwinski, A. H. (1970). Stochastic Processes and Filtering Theory. Academic Press, New York.

Kazakov, I. E. (1965). Generalization of the Method of Statistic Linearization to Multidimensional Systems. Avtom. & Telemekh., 26, 1210-1215.

Taylor, J. H. (1975). An Algorithmic State-Space/Describing Function Technique for Limit Cycle Analysis. T.O.M., The Analytic Sciences Corp. (TASC), Reading, MA. (Also issued as TASC TIM-612-1 to the Office of Naval Research, Oct. 1975).

Taylor, J. H. (1980). Applications of a General Limit Cycle Analysis Method for Multi-variable Systems. Part 2, Ch. 3 of Nonlinear System Analysis and Synthesis, Vol. 2 Applications, Ed. by Ramnath, R. V., Hedrick, J. K. and Paynter, H. M., publ. by the ASME.

Taylor, J. H. and Mohan, S. R. (1980). Describing Function Analysis of the Frequency Response of Highly Nonlinear Systems. Submitted to the IEEE Trans. on Auto. Control.

Taylor, J. H., Price, C. F., Siegel, J. and Gelb, A. (1980). Covariance Analysis of Nonlinear Stochastic Systems via Statistical Linearization. Part 2, Ch. 7 of Nonlinear System Analysis and Synthesis, Vol. 2 Applications, Ed. by Ramnath, R. V., Hedrick, J. K. and Paynter, H. M., publ. by the ASME.