Strictly Positive-Real Functions and the Lefschetz–Kalman–Yakubovich (LKY) Lemma

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A network made up of the lumped passive elements R, L, and C (resistance, inductance, and capacitance) has a driving point impedance $Z(s)$ that is rational and positive-real, and, conversely, any rational function $Z(s)$ that is positive-real can be realized as the driving point impedance of a passive RLC network. The properties of positive-real functions have thus been exhaustively studied in the evolution of modern network theory. Strictly positive-real functions have not received the same attention, however, and this deficiency has led to a basic lack of clarity in one area of absolute stability theory. A resolution of this difficulty as detailed in [1] is outlined in this letter.

Given $Z(s) = (n(s))/(d(s))$ having poles and zeros in the left half plane (Re $s \leqslant 0$), the necessary and sufficient condition that $Z(s)$ be positive-real (denoted $Z(s) \in \{PR\}$) is that Re $Z(i\omega) \geqslant 0$ for all real ω. The corresponding conditions for $Z(s)$ to be strictly positive-real ($Z(s) \in \{SPR\}$) have been given in two forms: $Z(s)$ must have poles and zeros in the open left half plane (Re $s < 0$) and either

$$\text{Re}\, \dot{Z}(i\omega) > 0, \qquad \omega \in (-\infty, \infty) \tag{1a}$$

or

$$\text{Re}\, Z(i\omega) \geqslant \delta > 0, \qquad \omega \in [-\infty, \infty]. \tag{1b}$$

Equation (1a) is not sufficiently strict [and this has led to a fundamental complication in the Lefschetz–Kalman–Yakubovich (LKY) lemma], while (1b) is too stringent.

First, the proposed definition of a strictly positive-real function is motivated by an appeal to network theory. A strictly positive-real function is said to correspond to the driving point impedance of a dissipative network, i.e., a network composed of resistors, lossy inductors, and lossy capacitors. These latter elements may be represented by $L(s + \epsilon)$ and $C(s + \epsilon)$, corresponding to an ideal inductor in series with $R_L = \epsilon L$ and an ideal capacitor in parallel with $G_C = \epsilon C$, respectively. Directly, an intuitively reasonable definition of strictly positive-real functions ensues.

Definition 1: $Z(s) \in \{SPR\}$ if and only if there exists some $\epsilon > 0$ such that $Z(s - \epsilon) \in \{PR\}$.//

Thus given any passive RLC network with $Z(s) \in \{PR\}$, a dissipative network is always obtained by substituting $L_i(s + \epsilon)$ and $C_i(s + \epsilon)$ for each $L_i s$ and $C_i s$ in $Z(s)$, yielding the driving point impedance $Z(s + \epsilon)$, and, conversely, for any $Z(s) \in \{SPR\}$ there must exist some $\epsilon_1 > 0$ such that $0 < \epsilon < \epsilon_1$ guarantees that $Z(s - \epsilon) \in \{SPR\}$ while $Z(s - \epsilon_1)$ is merely positive-real.

This definition leads to an important asymptotic property.

Corollary to Definition 1: If $Z(s) \in \{SPR\}$ then Re $Z(i\omega)$ can go to zero no more rapidly than ω^{-2} as $\omega \to \infty$.//

Proof: Given $Z(s) = (n(s))/(d(s)) \in \{SPR\}$; Re $Z(i\omega) \to 0$ as $\omega \to \infty$ only if order $[n(s)]$ = (order $[d(s)]$ − 1), i.e., if

$$Z(s) = \frac{c_n s^{n-1} + \cdots + c_2 s + c_1}{s^n + a_n s^{n-1} + \cdots + a_2 s + a_1}, \qquad \begin{array}{ll} c_i > 0, & i = 1, 2, \cdots, n \\ a_i > 0, & i = 1, 2, \cdots, n. \end{array}$$

By expansion,

$$\text{Re}\, Z(i\omega - \epsilon) = \text{Re}\, \frac{n(i\omega - \epsilon) d(-i\omega - \epsilon)}{d(i\omega - \epsilon) d(-i\omega - \epsilon)} = \frac{\omega^{2(n-1)} [a_n c_n - c_{n-1} - \epsilon c_n] + \cdots}{\omega^{2n} + \cdots}.$$

Thus since Re $Z(i\omega - \epsilon) \geqslant 0$ is to be satisfied as $\omega \to \infty$, clearly $(a_n c_n - c_{n-1}) \geqslant \epsilon c_n \triangleq b_n > 0$ is required, and so Re $Z(i\omega) \sim b_n/\omega^2$ as $\omega \to \infty$.//

Example (Guillemin [2]): Several points are clarified by considering the driving point impedance of a network made up of two parallel

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paths, the first a lossy capacitor [C in parallel with G] and the second a lossy inductor [L in series with R]. The normalized impedance is

$$Z_1(s) = \frac{s + c_1}{s^2 + a_2 s + a_1}, \qquad C = 1 \quad L = \frac{1}{a_1 - c_1(a_2 - c_1)}$$
$$G = (a_2 - c_1)C \qquad R = c_1 L.$$

Directly,

$$\text{Re}\, Z_1(i\omega) = \frac{(a_2 - c_1)\omega^2 + a_1 c_1}{(a_1 - \omega^2)^2 + (a_2\omega)^2}.$$

When $c_1 = a_2$, note that $G = 0$ and $\text{Re}\, Z_1(i\omega) \to 0$ as ω^{-4} when $\omega \to \infty$; if $c_1 = 0$ then $R = 0$. In both cases $Z_1(s)$ is only positive-real and the network having this driving point impedance cannot be realized with lossy elements. Only in the second case is $\text{Re}\, Z_1(i\omega)$ zero for finite ω (at $\omega = 0$); if $c_1 = a_2$ then $\text{Re}\, Z_1(i\omega) > 0$ for $\omega \epsilon (-\infty, \infty)$, so condition (1a) is not in itself a useful definition of a strictly positive-real function. Also, note that $Z_1(s)$ could not be accepted as a strictly positive-real function using condition (1b) under any circumstances.

Every real rational function that is proper (having no more zeros than poles) may be realized by a quadruple $\{\psi, c, A, b\}$ as

$$Z(s) = \psi + c^T (sI - A)^{-1} b \tag{2}$$

where ψ is a scalar, c and b are n-element column vectors and A is an $n \times n$ matrix. A fundamental network theoretic result (which is central to the solution of the absolute stability problem via the Lyapunov direct method) is the Kalman–Yakubovich lemma [3]. One form of this lemma, due to Lefschetz [4], is especially useful in the stability analysis of nonlinear time-varying systems (cf., Narendra and Taylor [5]).

Lemma 1: Given $\delta > 0$, a matrix A such that $|sI - A|$ has only zeros in the open left half plane, a real vector b such that (A, b) is completely controllable, a real vector c, a scalar ψ, and an arbitrary real symmetric positive definite matrix L ($L = L^T > 0$); then a real vector q and a real matrix $P = P^T > 0$ satisfying

$$A^T P + PA = -qq^T - \delta L \tag{3a}$$
$$Pb - c = \sqrt{2\psi}\, q \tag{3b}$$

exist if and only if δ is sufficiently small and $Z(s) \in \{SPR\}$.//

Only the constraint $\text{Re}\, Z(i\omega) > 0$ was originally required in [4]. In Lefschetz, Meyer, and Wonham [6] it was pointed out that this condition is too lax if $\psi = 0$; in that case, the additional requirement $c^T Ab < 0$ must be imposed. Using phase variable canonical form (as in [4], with no loss in generality), viz.,

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ \\ c_n \end{bmatrix} \qquad A = \left[\begin{array}{c|c} 0 & \\ 0 & \\ \cdot & \\ \cdot & I \\ \cdot & \\ 0 & \\ \hline -a_1 & -a_2 \cdots -a_n \end{array}\right] \qquad b = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

direct expansion results in

$$Z(s) = \psi + \frac{c_n s^{n-1} + \cdots + c_2 s + c_1}{s^n + a_n s^{n-1} + \cdots + a_2 s + a_1}$$

and $c^T Ab = (c_{n-1} - a_n c_n)$. Thus

$$\text{Re}\, Z(i\omega) = \psi + \frac{(-c^T Ab)\,\omega^{2(n-1)} + \cdots}{\omega^{2n} + \cdots}$$

and the condition $Z(s) \in \{SPR\}$ immediately guarantees that $c^T Ab < 0$ if $\psi = 0$ by the Corollary to Definition 1. Hence Definition 1 obviates the necessity of introducing the seemingly artificial auxiliary condition $c^T Ab < 0$ explicitly.

Finally, Lemma 1 and Definition 1 make the LKY lemma entirely equivalent to one form of the Kalman-Yakubovich lemma due to Meyer [7]. Given $Z(s) \in \{SPR\}$, define

$$\hat{Z}(s) \triangleq Z(s - \epsilon) = \psi + c^T[(s - \epsilon)I - A]^{-1} b \triangleq \psi + c^T(sI - \hat{A})^{-1} b$$

where $\hat{A} \triangleq A + \epsilon I$; for $\epsilon > 0$ sufficiently small, $\hat{Z}(s) \in \{PR\}$, and $|sI - \hat{A}|$ has zeros only in the open left half plane.

Lemma 2 (Meyer [7, Lemma 1]): Given a matrix $\hat{A}$ such that $|sI - \hat{A}|$ has only zeros in the open left half plane, a real vector b such that $(\hat{A}, b)$ is completely controllable, a real vector c, and a scalar ψ; then a real vector $\hat{q}$, a real symmetric positive semidefinite matrix M ($M = M^T \geq 0$), and a real matrix $P = P^T > 0$ satisfying

$$\hat{A}^T P + P\hat{A} = -\hat{q}\hat{q}^T - M \tag{4a}$$
$$Pb - c = \sqrt{2\psi}\, \hat{q} \tag{4b}$$

exist if and only if $\hat{Z}(s) \in \{PR\}$.//

Substituting $\hat{A} = A + \epsilon I$ into (4a) yields

$$A^T P + PA = -\hat{q}\hat{q}^T - (M + 2\epsilon P).$$

Since $(M + 2\epsilon P)$ is symmetric and positive definite, an elementary result of matrix theory is that for any $L = L^T > 0$ there exists a $\delta > 0$ such that

$$M + 2\epsilon P = \delta L + \hat{M}$$

and $\hat{M} = \hat{M}^T \geq 0$. From Meyer's proof (it is not entirely obvious here) it is thus always possible to satisfy (3a) with q satisfying $qq^T = \hat{q}\hat{q}^T + \hat{M}$.

This new definition of strictly positive-real functions may not be universally useful in every situation where it is necessary to impose a stronger condition than $Z(s) \in \{PR\}$; however, these points demonstrate that it plays an important role in the present context.

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