SIAM J. APPL. MATH.
Vol. 18, No. 2, March 1970

THE CORDUNEANU-POPOV APPROACH TO THE STABILITY OF NONLINEAR TIME-VARYING SYSTEMS*

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Abstract. Previous extensions of the stability criterion of Popov [1] for systems with a stable linear time-invariant plant $G(s)$ followed by a nonlinear time-varying gain $k(t)f(\cdot)$ entailed the definition of nonlinearity classes $[f(\cdot) \in N]$ and corresponding frequency domain multipliers $Z_N(s)$. Then, defining a measure of nonlinearity

$$F_{\min} \equiv \min_x \left\{ \frac{xf(x)}{\int_0^x f(z)\, dz} \right\},$$

stability is ensured if $Z(s) \in Z_N(s)$ exists, such that (i) $G(s)Z(s)$ is strictly positive real, (ii) $Z(s - \Lambda) \in Z_N(s)$ and (iii)

$$\frac{1}{k}\frac{dk}{dt} \leqq \Lambda F_{\min} \qquad \text{(see [5]).}$$

In this paper, the point by point requirement of (iii) is replaced by an integral criterion. In many cases the constraints on $k(t)$ predicated by the integral inequality are substantially less strict than those of (iii) above. As (iii) is a special case of the integral relation, the results are never more strict.

The development here will be limited to the two most general classes of nonlinearities, F (first and third quadrant nonlinearities, introduced by Lur'e and Postnikov [15] and considered by Popov [1]) and FM (the subclass of F consisting of monotonically increasing functions); for more restricted classes the details are identical. For a complete exposition, see [13].

1. Introduction and system description. The behavior of the system under consideration is described by the n-dimensional state vector differential equation

$$\begin{aligned} \dot{x} &= Ax - bk(t)f(\sigma_0), \\ \sigma_0 &= h^T x, \end{aligned} \tag{1.1}$$

where the linear time-invariant system $\dot{x} = Ax - bk_0h^Tx$, $0 < k_0 < \infty$, k_0 an arbitrary constant, is completely controllable and completely observable, and A is an asymptotically stable matrix. With no loss in generality [10], the phase-variable canonical form may be used:

$$h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}, \quad A = \left[\begin{array}{c|ccc} 0 & & & \\ 0 & & I & \\ \vdots & & & \\ 0 & & & \\ \hline -a_1 & -a_2 & \cdots & -a_n \end{array}\right], \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{1.2}$$

It is further assumed that $0 \leqq k(t) < \infty$ for all t, and that $k(t)$ is absolutely continuous, thus ensuring the existence of dk/dt almost everywhere.

* Received by the editors December 13, 1968, and in revised form April 28, 1969.

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The nonlinearity $f(z)$ is continuous and satisfies $f(0) = 0$. It will be assumed to belong to one of the following classes:

$$\text{(1.3)}\quad \begin{aligned} &\text{(i)} && f(z) \in F \quad \text{if } 0 < \frac{f(z)}{z} < \infty \quad \text{for all } z \neq 0, \\ &\text{(ii)} && f(z) \in FM \quad \text{if } f(z) \in F \quad \text{and} \quad \frac{df(z)}{dz} \geqq 0 \quad \text{for all } z. \end{aligned}$$

As a measure of nonlinearity, it is useful to introduce the parameter $F_{\min}$:

$$\text{(1.4)}\qquad F_{\min} = \min_x \{F(x)\} \equiv \min_x \left\{ \frac{xf(x)}{\int_0^x f(z)\,dz} \right\}.$$

By inspection, it can be seen that the following ranges for $F_{\min}$ are permitted:

(i) F: $0 < F_{\min} < \infty$,

(ii) FM: $1 < F_{\min} < \infty$.

For linear systems, $F_{\min} = 2$. The more information one has about the nonlinearity, the more precisely (less conservatively) $F_{\min}$ may be determined.

An alternative representation of the system of equation (1.1) is defined in Fig. 1 where clearly

$$\text{(1.5)}\qquad G(s) = h^T(sI - A)^{-1}b$$

and, by substituting (1.1),

$$\text{(1.6)}\qquad G(s) = \frac{h_n s^{n-1} + \cdots + h_2 s + h_1}{s^n + a_n s^{n-1} + \cdots + a_2 s + a_1}.$$

Observability ensures that none of the poles and zeros of $G(s)$ cancel.

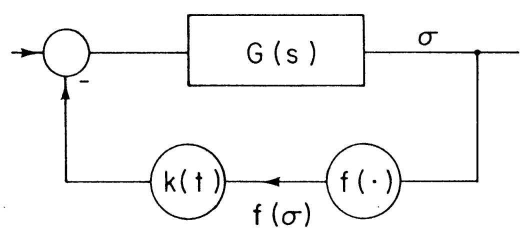

FIG. 1

In this paper, the stability of systems of the form defined above is considered. In an earlier work [5] a generalized Lur'e–Postnikov function V, equation (2.1), was used and the stability conditions outlined in the abstract were derived by demanding that $\dot{V}$ (the time derivative of V evaluated along the system trajectory) be always negative. Here the requirement that $\dot{V} < 0$ for all t is replaced by $\dot{V} \leqq g(t)V$, where $\dot{y} = g(t)y$ is a stable differential equation. That this is sufficient to ensure stability has been proven by Corduneanu; refer for instance to Hahn [6] or Sansone and Conti [7].

2. Form of the Lyapunov function and frequency domain multipliers. The Lyapunov function used in this paper is identical to that of [5], viz.

$$V(x, t) = \tfrac{1}{2}x^T P x + \sum_{i=0}^{m} \beta_i k(t) \int_0^{\sigma_i} f(z)\,dz, \qquad \beta_0 \equiv 1, \tag{2.1}$$

where P is an $n \times n$ symmetric positive definite matrix $[P = P^T > 0]$, all $\beta_i > 0$, and the signals used in the upper limits of the integral terms are derived from the state vector by the relations

$$\sigma_i = r_i^T x, \qquad r_0 \equiv h. \tag{2.2}$$

Obviously $V(x, t)$ is positive definite and decrescent since $k(t)$ is nonnegative and bounded and $f(\sigma)$ lies in the first and third quadrants.

The original Lur'e–Postnikov form $\frac{1}{2}x^T P x + \int_0^{\sigma_0} f(z)\,dz$ was used by Popov [1] and the use of more than one signal (multiple integrals) was introduced by Narendra and Neuman [2]. The use of (2.1) was original with [5].

Making no further assumptions about the signals $r_i^T x$ other than $r_0 = h$, one sees that

$$\begin{aligned}\dot{V} = {} & \tfrac{1}{2}x^T(PA + A^T P)x - k(t)f(\sigma_0)x^T[Pb - (\lambda_0 h + A^T h)] - h^T b[k(t)f(\sigma_0)]^2 \\ & - \lambda_0 k(t)\sigma_0 f(\sigma_0) + \sum_{i=1}^{m} \beta_i k(t) f(\sigma_i) r_i^T [Ax - bk(t)f(\sigma_0)] \\ & + \sum_{i=0}^{m} \beta_i \frac{dk}{dt} \int_0^{\sigma_i} f(z)\,dz.\end{aligned} \tag{2.3}$$

To obtain this expression, $\lambda_0 k(t)\sigma_0 f(\sigma_0)$ is added to and subtracted from the formal time derivative of V evaluated along system trajectories, i.e., $\dot{V} = (\nabla V)^T \dot{x} + \partial V/\partial t$.

By a judicious choice of signals $(r_i^T x)$, one may obtain $\dot{V} \leqq g(t)V$, and the theorem of Corduneanu may be applied.

In order to make the formulation of the stability theorem more compact, the following frequency domain multipliers are defined:

(A) $f(\sigma) \in F$: for this general class of nonlinearities, one is restricted to the use of the Popov multiplier.

$$Z_F(s) = s + \lambda_0, \tag{2.4}$$

where $\lambda_0 \geqq 0$. Clearly $\Lambda = \lambda_0$, i.e., $Z(s - \lambda) \in Z_F$ for any $\lambda \leqq \Lambda = \lambda_0$.

(B) $f(\sigma) \in FM$: by constraining $f(\sigma)$ to be monotonic, it is possible to use a more general RL multiplier,

$$Z_{FM}(s) = (s + \lambda_0) + \sum_{i=1}^{m_1} \gamma_i \frac{s + \lambda_0}{s + \eta_i}, \tag{2.5}$$

where $0 \leqq \lambda_0 < \eta_i$, $i = 1, 2, \cdots, m_1$. It can be shown that any general RL impedance with poles at $s = -\eta_i$ may be expanded into this form. The phase of this multiplier, as that of the Popov multiplier, must be in the range $(0, 90°)$, but it no longer needs to increase monotonically with frequency as in the case of $(s + \lambda_0)$. Again it is evident that $\Lambda = \lambda_0$.

For several further restricted classes of nonlinearities, more general multipliers have been derived in [5]. In particular if one considers the subclass of FM consisting of odd functions $[f(-z) = -f(+z); f \in FMO]$ an RLC multiplier with poles in the sector $135° < \arg(\text{pole}) < 225°$ may be used, and if the function is an odd monotonic power law $[f \in FMOP]$ this sector may be enlarged to $116.6° < \arg(\text{pole}) < 243.4°$, that is, the imaginary part of the pole may be nearly twice its real part. If the system is linear, then an LC multiplier may be used.

In this paper it will be assumed that *poles of* $Z(s)$ *are zeros of* $G(s)$ so that the maximum number of RL terms is less than n. While other methods of proof might obviate this assumption, it may be seen that in the application of the theorem it is practical to attempt pole-zero cancellation to simplify the determination of the positive realness of $G(s)Z(s)$.

3. Stability theorems. The proof of the theorems given here will be treated in a subsequent section of this paper. A discussion of the theorems will follow directly.

THEOREM 1. *If the system of* § 2 *satisfies*

(i) $G(s - \rho)$ *is an asymptotically stable transfer function,* $\rho \geqq 0$,

(ii) $f(\sigma) \in N$,

(iii) $Z \in Z_N(s)$ *exists such that*

(a) $G(s - \rho)Z(s - \rho)$ *is positive real,*

(b) $Z(s - \Lambda) \in Z_N(s)$,

$$g(t) \equiv \sup_t \left\{ -2\rho\, ; -\left(\Lambda F_{\min} - \frac{1}{k}\frac{dk}{dt} \right) \right\} \tag{3.1 (iv)}$$

satisfies

$$\int^{\infty} g(t)\, dt = -\infty \tag{3.2}$$

and, if the system is nonlinear, $(1/k)dk/dt$ *is continuous*[1] *then the system is absolutely stable.*

It is evident that as $\rho \to 0$, the above theorem requirements become those of previous work [5], and, in particular, (iv) becomes the point by point criterion $(1/k)\, dk/dt \leqq \Lambda F_{\min}$.

If $k(t)$ is periodic, i.e., $k(t) + k(t + T)$ for all t, then the integral relation of (iv) becomes $\int_0^T g(t)\, dt < 0$.

A special case for linear time-varying systems is found to be useful because the stability conditions obviate the use of multipliers (although an LC multiplier is required for the proof).

THEOREM 2. *Define* $H(j\omega) \equiv G(j\omega - \rho)$, $\rho \geqq 0$. *If the system of* § 2 *satisfies*

(i) $f(\sigma) = \sigma$ *and* $H(j\omega)$ *is asymptotically stable,*

(ii) $H/(1 + K_0 H)$ *is asymptotically stable for all* $K_0 \in [0, \infty)$,

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[1] If the system is linear time-varying, continuity is not required; see footnote 3.

(iii)
$$\begin{aligned}\operatorname{Im} H(j\omega) &< 0, \quad \omega \in (0, \omega_1),\\ H(j\omega_1) &= 0,\\ \operatorname{Im} H(j\omega) &> 0, \quad \omega \in (\omega_1, \omega_2),\\ H(j\omega_2) &= 0,\\ &\vdots \quad ,\end{aligned}$$

where $0 < \omega_1 < \omega_2 \cdots \leqq \infty$,

(iv) $k(t) \in [0, \infty)$; $k(t) = k(t + T)$ *for all t and*

$$I \equiv \frac{1}{T}\int_0^T \frac{|\dot{k}|}{k}\,dt < 4\rho, \tag{3.3}$$

then it is absolutely stable.

This theorem is a special case of a theorem due to Freedman and Zames [9]. In the latter work, $G(s)$ need not be a ratio of polynomials in "s" and (iii) is not necessary. The restriction (iii) is required here because poles of the LC multiplier used in the proof must be zeros of $H(s)$. This theorem is thus included not for its novelty but to demonstrate the intimate relationship between it and the more general main theorem.

There is one shortcoming to Theorem 2 that is shared by the more general formulation due to Freedman and Zames. It is sometimes possible that the point by point requirement $(1/k)(dk/dt) < 2\Lambda$ may yield a better stability bound than the integral criterion. The main theorem properly applied will lead ineluctably to the best result whereas Theorem 2 will not. This will be demonstrated in Example 2.

The principle strength of the main theorem is, of course, its applicability to *nonlinear* time-varying systems.

4. Examples. Two applications of the theorems are included to clarify some of the comments made in earlier sections. The first example will demonstrate a meaningful case in which the new approach has distinct advantages over the former point by point criterion. The second problem shows how the main theorem avoids the shortcoming of Theorem 2.

Example 1. Consider a damped nonlinear form of the Mathieu equation:

$$(\ddot{x} + 2\zeta\dot{x} + \zeta^2 x) + [(1 - \zeta^2) - \varepsilon\cos(2t)]f(x) = 0. \tag{4.1}$$

The stability boundaries in the parameter space $\{a, q\}$ for $\zeta = 0$ and $f(x) = ax$, $q \equiv \frac{1}{2}a\varepsilon$ are well known [11]. In the damped linear case ($\zeta \neq 0, f(x) = ax$) a number of approximate boundaries have been published; see [12] for some of these results.

In state vector form (4.1) corresponds to

$$A = \begin{bmatrix} 0 & 1 \\ -\zeta^2 & =2\zeta \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h = \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$

therefore $G(s) = h^T(sI - A)^{-1}b = 1/(s^2 + 2\zeta s + \zeta^2)$.

We show (i) i.e., $G(s - \rho)$ must be asymptotically stable.

$$G(s - \rho) = \frac{1}{s^2 + (2\zeta - 2\rho)s + (\zeta - \rho)^2}.$$

For stability, one requires $\rho < \zeta$; therefore define

$$\rho \equiv (1 - \alpha)\zeta, \qquad 0 < \alpha \leqq 1. \tag{4.2}$$

For (ii) and (iii), $f(\sigma) \in \mathrm{F}$; therefore inspect the positive realness of

$$H(s) \equiv \frac{s + \lambda - \rho}{s^2 + (2\zeta - 2\rho)s + (\alpha\zeta)^2}.$$

It is not difficult to show that $(s + a)/(s^2 + bs + c)$ is positive real for all $a \leqq b$; therefore allow λ to be as large as possible to maximize Λ:

$$\begin{aligned}(\lambda - \rho) &= (2\zeta - 2\rho),\\ \Lambda = \lambda &= (1 + \alpha)\zeta.\end{aligned} \tag{4.3}$$

For simplicity, ζ will be assumed to be small ($\zeta \ll 1$). Since it will be seen that for $F_{\min} \leqq O(1)$, $\varepsilon \leqq O(\zeta)$, $\varepsilon \ll 1$ also.

We show (iv), i.e., $g(t) = \sup\{-2\rho, -(\Lambda F_{\min} - (1/k)(dk/dt)\}$ must satisfy $\int_0^T g(t)\,dt < 0$, as follows: One has

$$\frac{1}{k}\frac{dk}{dt} = \frac{2\varepsilon \sin(2t)}{1 - \zeta^2 - \varepsilon\cos(2t)} \approx 2\varepsilon\sin(2t),$$

since ε and ζ are small. Thus $g(t) = \sup\{-2(1 - \alpha)\zeta, -[(1 + \alpha)\zeta F_{\min} - 2\varepsilon\sin(2t)]\}$. To obtain the former upper bound on ε (using the stability theorem of [5]), let $\rho \to 0$ (or $\alpha \to 1$); then require that

$$f(t) \equiv (1 + \alpha)\zeta F_{\min} - 2\varepsilon\sin(2t) \geqq 0 \qquad \text{for all } t.$$

As $\alpha \to 1$, one needs

$$\varepsilon < \varepsilon^* \equiv \zeta F_{\min}. \tag{4.4}$$

In Fig. 2, $g(t)$ (solid curve) is drawn for $\varepsilon = \zeta F_{\min}$ and $F_{\min} \approx 2$. It can be seen that $\int_0^T g(t)\,dt$ is quite negative, and hence a larger value of ε could be used with stability still assured.

The actual computation of the maximum value of ε^{**} for stability for each value of $F_{\min}$ is sufficiently complex that only the result in the form of a graph of ε^{**} versus $F_{\min}$ will be given here Fig. 3. The complete analysis is to be found in the Appendix. Note that for $F_{\min} = 2$ ($f(\sigma) = a\sigma$ being a special case) an improvement of 57% is achieved, and the gain increases dramatically as $F_{\min} \to 0$.

Example 2. Consider $G(s) = 1/(s^2 + 2\zeta s + \zeta^2)$, $\zeta \ll 1$, $f(\sigma) = \sigma$ and for $n = 0, 1, \cdots$,

$$k(t) = \begin{cases} e^{at}, & nT \leqq t < (nT + t_1) \equiv nT + \dfrac{1}{a}\log(1 + \varepsilon),\\[2ex] e^{4a(T-t)}, & (nT + t_1) \leqq t < (n + 1)T \equiv nT + \dfrac{5}{4a}\log(1 + \varepsilon).\end{cases} \tag{4.5}$$

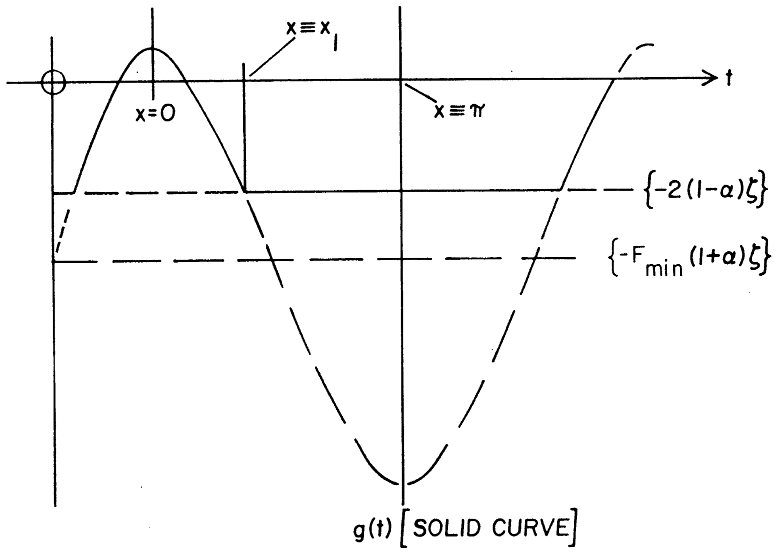
x≡x1
t
x=0
x≡π
{-2(1-α)ζ}
{-Fmin(1+α)ζ}
g(t) [SOLID CURVE]

FIG. 2

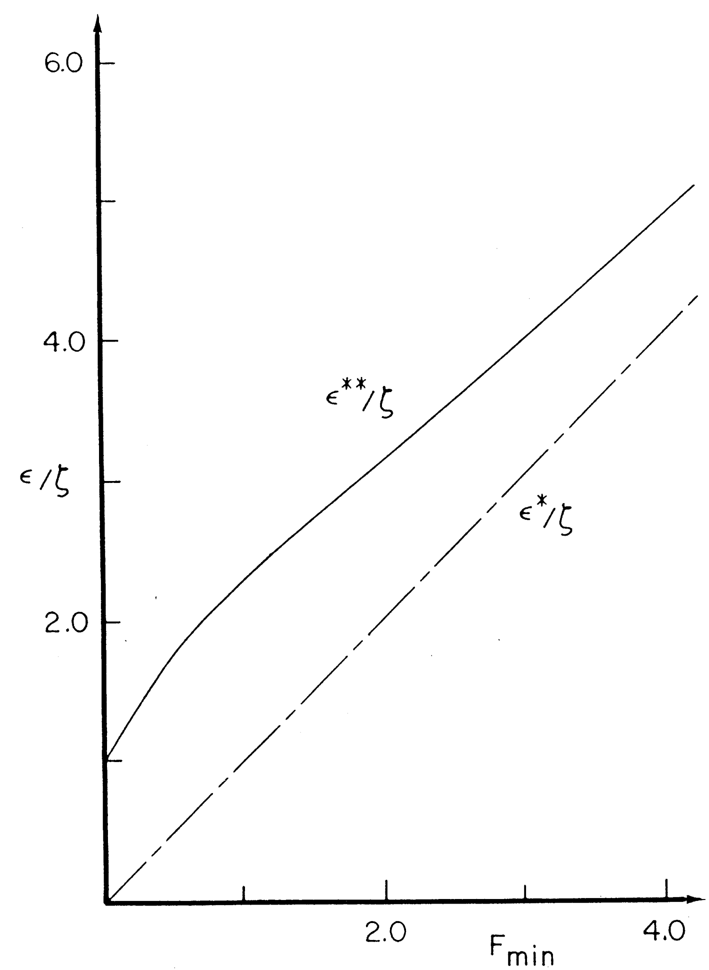
6.0
4.0
2.0
ε/ζ
ε**/ζ
ε*/ζ
2.0
4.0
Fmin

FIG. 3

Thus

$$\frac{1}{k}\frac{dk}{dt} = \begin{cases} +a, & nT \leqq t < (nT + t_1), \\ -4a, & (nT + t_1) \leqq t < (n + 1)T. \end{cases}$$

Apply Theorem 2: $G(s-\rho)$ is stable for all $\rho \in [0, \zeta)$, and $G(s-\rho)/(1+K_0G(s-\rho))$ is stable for all $\rho \in [0, \zeta)$, $K_0 \in [0, \infty)$. Furthermore $\mathrm{Im}\, H(j\omega) < 0$ for all $\omega \in (0, \infty)$. By inspection of (3.3),

$$I = \frac{1}{T}[at_1 + 4a(T - t_1)];$$

by direct substitution from (4.5), $I = 8a/5 < 4\zeta$ is the final stability requirement. Thus, the above system is stable if $a < 5\zeta/2$.

Now an application of Theorem 1 will be made to demonstrate that a higher upper bound on a may be obtained corresponding to the point by point criterion $(\rho \to 0)$. As in Example 1, choose $\rho = (1 - \alpha)\zeta$, so $\lambda_0 = \Lambda = (1 + \alpha)\zeta$, where $0 < \alpha \leqq 1$. Hence $g(t)$, (equation (4.1)) may be seen to be as follows:

$$g(t) = \begin{cases} a - 2(1 + \alpha)\zeta, & 0 \leqq t < t_1, \\ -2(1 - \alpha)\zeta, & t_1 \leqq t < T. \end{cases}$$

Thus $I \equiv \int_0^T g(t)\, dt$ is

$$\begin{aligned} I &= [a - 2(1 + \alpha)\zeta]t_1 - 2(1 - \alpha)(T - t_1) \\ &= \frac{1}{a}\log(1 + \varepsilon)\left[a - \left(\frac{5}{2} + \frac{3}{2}\alpha\right)\zeta\right]; \end{aligned}$$

hence the integral is negative if $a < (5/2 + 3\alpha/2)\zeta$. If $\alpha = 0$ is chosen, the previous upper bound (by Theorem 2) is obtained, but, obviously, as $\alpha \to 1 (\rho \to 0)$ the bound increases to 4ζ.

In this simple example, it could be seen by inspection that the point by point criterion would yield the less strict upper bound on a. If $k(t)$ were more complex, however, it would not be so clear. As demonstrated above, the main theorem takes this into account with little added complexity.

5. Basic lemmas.

LEMMA 1. *For all FM nonlinearities,*

$$(\sigma_1 - \sigma_2)[f(\sigma_1) - f(\sigma_2)] \geqq 0 \qquad \textit{for all } \sigma_1, \sigma_2.$$

Proof. The result follows by definition, since $df(\sigma)/d\sigma \geqq 0$ for all σ is $f \in FM$.

LEMMA 2. *If* $G(s)|_{s=-\eta} = 0$, *then the vector* c, $c^T \equiv h^T(\eta I + A)^{-1}$, [2]*has the properties*

(i) $$c^T(sI - A)^{-1}b = \frac{G(s)}{s + \eta},$$

(ii) $$c^Tb = 0,$$

(iii) $$c^TAx = h^Tx - \eta c^Tx.$$

[2] Zeros of the numerator of $G(s)$ must not be eigenvalues of A in order for $(\eta I + A)^{-1}$ to exist. Recall that the observability of the system ensures that no zeros of G cancel with poles.

Proof. See [5].

LEMMA 3. *If* $G(s) = h^T(sI - A)^{-1}b$, *then* $sG(s) = h^Tb + h^TA(sI - A)^{-1}b$.

Proof. $[h^Tb + h^TA(sI - A)^{-1}b] = h^T[(sI - A) + A](sI - A)^{-1}b = sG(s)$.

LEMMA 4. *Given* $A^* \equiv A + \rho I$ *an asymptotically stable matrix, vectors* $b \neq 0$ *and* k, *and a scalar* $\tau \geqq 0$, *then a necessary and sufficient condition for the existence of a solution to*

$$\text{(a)} \qquad A^TP + PA = -qq^T - 2\rho P - D,$$

$$\text{(b)} \qquad Pb - k = \sqrt{\tau}q,$$

where $P > 0$ *and* $D \geqq 0$ *necessarily is that*

$$\text{(c)} \qquad \tau + 2\operatorname{Re}\{k^T(sI - A)^{-1}b\}|_{s=-\rho+j\omega} \geqq 0 \qquad \text{for all real } \omega.$$

Proof. See [3], [4]; this modification follows trivially.

6. Theorem proofs. Consider the case of $f(\sigma) \in FM$: For each of the m_1 real zeros of $G(s)$, use the following as Lyapunov signals:

$$\sigma_i = \frac{\gamma_i}{\beta_i}h^T(\eta_iI + A)^{-1}x. \tag{6.1}$$

By Lemma 2, note that $r_i^Tb = 0$ and $r_i^TAx = (1/\beta_i)(\gamma_i\sigma_0 - \beta_i\eta_i\sigma_i)$. Substituting directly in $\dot{V}$, (2.3) yields

$$\begin{aligned}\dot{V} = {} & \tfrac{1}{2}x^T(PA + A^TP)x - h^Tb[k(t)f(\sigma_0)]^2 - \lambda_0k(t)\sigma_0f(\sigma_0) \\ & -k(t)f(\sigma_0)x^T[Pb - (\lambda_0h + A^Th) - \sum_{i=1}^{m_1}\gamma_i(h - r_i)] \\ & - \sum_{i=1}^{m_1}\gamma_ik(t)(\sigma_0 - \sigma_i)[f(\sigma_0) - f(\sigma_i)] \\ & - \sum_{i=1}^{m_1}(\beta_i\eta_i - \gamma_i)k(t)\sigma_if(\sigma_i) + \sum_{i=0}^{m_1}\beta_i\frac{dk}{dt}\int_0^{\sigma_i}f(z)\,dz.\end{aligned}$$

This form was obtained by adding and subtracting the term

$$\sum_{i=1}^{m_1}\gamma_ik(t)(\sigma_0 - \sigma_i)[f(\sigma_0) - f(\sigma_i)],$$

which is nonnegative for all FM nonlinearities by Lemma 1. Recall the definition of $F(\sigma)$, (1.4), choose β_i large enough that

$$\beta_i\eta_i - \gamma_i \equiv \varepsilon_i > 0 \tag{6.2}$$

and use the Meyer–Kalman–Yakubovich lemma (Lemma 4) to obtain

$$\begin{aligned}\dot{V} = {} & -\tfrac{1}{2}[q^Tx + \sqrt{2h^Tb}\,kf(\sigma_0)]^2 - x^TDx - \sum_{i=1}^{m_1}\gamma_ik(t)(\sigma_0 - \sigma_i)[f(\sigma_0) - f(\sigma_i)] \\ & - \int_0^{\sigma_0}f(z)dz\left[\lambda_0kF(\sigma_0) - \frac{dk}{dt}\right] - \sum_{i=1}^{m_1}\beta_i\int_0^{\sigma_i}f(z)dz\left[\frac{\varepsilon_i}{\beta_i}kF(\sigma_i) - \frac{dk}{dt}\right] - \rho x^TPx,\end{aligned} \tag{6.3}$$

provided that the Kalman relation

$$h^Tb + \mathrm{Re}\left[\left\{(A^Th + \lambda_0 h) + \sum_{i=1}^{m_1} \gamma_i(h - r_i)\right\}^T (sI - A)^{-1}b\right] \geqq 0, \qquad s = -\rho + j\omega, \tag{6.4}$$

is satisfied for all real ω.

By Lemma 2, Lemma 3 and (6.2), this is equivalent to the requirement that $G(s - \rho)Z_{FM}(s - \rho)$ be positive real, where

$$Z_{FM}(s) = (s + \lambda_0) + \sum_{i=1}^{m_1} \gamma_i \frac{s + \varepsilon_i/\beta_i}{s + \eta_i}. \tag{6.5}$$

By inspection of (6.2), $0 < \varepsilon_i/\beta_i < \eta_i$, so it is evident that (6.5) is the expansion of an arbitrary RL function. It has been shown [5, Appendix] that, without any loss of generality, one may set $\varepsilon_i/\beta_i = \lambda_0 \equiv \Lambda$, where $\lambda_0 < \eta_i$, $i = 1, 2, \cdots, m_1$ (λ_0 is the smallest zero of Z_{RL}). One then obtains

$$\dot{V} \leqq -2\rho\left(\frac{1}{2}x^TPx\right) - \sum_{i=0}^{m_1} \beta_i k(t) \int_0^{\sigma_i} f(z)\,dz\left[\Lambda F(\sigma_i) - \frac{1}{k}\frac{dk}{dt}\right]. \tag{6.6}$$

Clearly, if $g(t) \equiv \sup\{-2\rho; -[\Lambda F_{\min} - (1/k)\,dk/dt]\}$, it is easily demonstrated that $\dot{V} \leqq g(t)V$; $g(t)$ has been chosen to be as negative as possible under this constraint.

At this point, apply Corduneanu's theorem [6], [7].

Let $\phi(r, t)$ be continuous for $r \in [0, h]$ and $t \in [t_0, \infty)$, and of such a nature that the scalar differential equation $\dot{y} = \phi(y, t)$ has solutions in the above domain (one solution being $y \equiv 0$). Let $v(x, t)$ be positive definite and let $\dot{v} \leqq \phi(v(x, t), t)$. If $y \equiv 0$ is stable (asymptotically stable), the same holds for the equilibrium of $\dot{x} = f(x, t)$.

In this case, $\phi(y, t) = g(t)y$, which satisfies all the conditions above if $g(t)$ is continuous. Since the solution of $\dot{y} = g(t)y$ is $y = \alpha \exp \int_0^t g(t)\,dt$, the requirement that $\int^{\infty} g(t)\,dt = -\infty$ is sufficient for the absolute stability of the system in question, completing the proof. ($g(t)$ is continuous unless ($\dot{k}/k$) is discontinuous; this is not a problem in meaningful physical systems[3].)

The proof may be seen to be a simple extension of the previous proof [5], thus it would be unnecessarily repetitive to carry out the proofs for $f(\sigma) \in FMO$, $FMOP$ or $f(\sigma) = \sigma$. For $f(\sigma) \in F$, simply use no signals other than σ_0 in the above development.

In proving Theorem 2, consider $H(j\omega) \equiv G(j\omega - \rho)$ satisfying the conditions of Theorem 2. It is evident that the LC function $Z^*(s) = (s(s^2 + \omega_2^2) \cdots)/((s^2 + \omega_1^2) \cdots)$ will render $H(s)Z^*(s)$ strictly positive real. This may be expanded into

$$Z^*(s) = s + \sum_i \delta_i \frac{s}{s^2 + \omega_i^2}, \qquad \delta_i > 0,$$

[3] In a linear time-varying situation, only $k(t)$ need be continuous [14, Theorem 10.3].

where ω_i are those frequencies that are poles of Z^*. Recall that $H(s)Z^*(s) = G(s - \rho)Z(s - \rho)$ so that

$$Z(s) = (s + \rho) + \sum_i \delta_i \frac{s + \rho}{(s + \rho)^2 + \omega_i^2},$$

and by inspection $\Lambda = \rho$, i.e., $Z(s - \rho)$ is an LC function.

Thus the only provision of the main theorem still to be met is that

$$g(t) \equiv \sup\left\{-2\rho; -\left[2\rho - \frac{\dot{k}}{k}\right]\right\}$$

satisfying

$$\int_0^T g(t)\,dt < 0.$$

Since $k(t)$ is periodic, $\log k(t)$ is also, hence one may expand $\log k(t) = a_0 + \sum_{i=1}^{\infty} a_i \sin(2\pi t/T + \phi_i)$. By inspection $(1/k)(dk/dt) = (d/dt)\{\log k(t)\}$ has a zero mean and therefore

$$\frac{1}{T}\int_0^T \left(\frac{1}{k}\frac{dk}{dt}\right) dt = 0. \tag{6.7}$$

DEFINITIONS.

(i) $$\tau_p \equiv t \in [0, T], \quad \text{such that } \frac{1}{k}\frac{dk}{dt} \geqq 0,$$

(ii) $$\tau_N \equiv t \in [0, T], \quad \text{such that } \frac{1}{k}\frac{dk}{dt} < 0,$$

(iii) $$T_p \equiv \int_{\tau_p} 1 \cdot dt,$$

(iv) $$T_N \equiv \int_{\tau_N} 1 \cdot dt.$$

Note that τ_p and τ_N are those subintervals within $[0, T]$ when $(1/k)(dk/dt)$ is positive and negative, respectively, and T_p and T_N are the measures of each set of subintervals. Clearly $T = T_p + T_N$. As a consequence of (6.7) and these definitions,

$$\frac{1}{T}\int_0^T \left(\frac{1}{k}\frac{dk}{dt}\right) dt = \frac{1}{T}\left\{\int_{\tau_p}\left(\frac{1}{k}\frac{dk}{dt}\right) dt + \int_{\tau_N}\left(\frac{1}{k}\frac{dk}{dt}\right) dt\right\} = 0,$$

so that

$$\int_{\tau_N}\left(\frac{1}{k}\frac{dk}{dt}\right) dt = -\int_{\tau_p}\left(\frac{1}{k}\frac{dk}{dt}\right) dt. \tag{6.8}$$

By definition

$$g(t) = \begin{cases} -2\rho, & t \in \tau_N, \\ -2\rho + \dfrac{1}{k}\dfrac{dk}{dt}, & t \in \tau_p. \end{cases}$$

Hence

$$\int_0^T g(t)\,dt = \int_{\tau_N} (-2\rho)\,dt + \int_{\tau_p} \left(-2\rho + \frac{1}{k}\frac{dk}{dt}\right) dt$$

$$= -2\rho[T_p + T_N] + \int_{\tau_p} \left(\frac{1}{k}\frac{dk}{dt}\right) dt,$$

so that the requirement $\int_0^T g(t)\,dt < 0$ is equivalent to

$$\text{(6.9)} \qquad \frac{1}{T}\int_{\tau_p} \left(\frac{1}{k}\frac{dk}{dt}\right) dt < 2\rho.$$

The proof is completed in noting that

$$\frac{1}{T}\int_0^T \left|\frac{1}{k}\frac{dk}{dt}\right| dt = \frac{1}{T}\left\{\int_{\tau_p}\left(\frac{1}{k}\frac{dk}{dt}\right) dt - \int_{\tau_N}\left(\frac{1}{k}\frac{dk}{dt}\right) dt\right\} = \frac{2}{T}\int_{\tau_p}\left(\frac{1}{k}\frac{dk}{dt}\right) dt,$$

by (6.8).

Appendix. From the preliminary development of Example 1, $g(t) = \sup_t \{-2(1-\alpha)\zeta, -[(1+\alpha)\zeta F_{\min} - 2\varepsilon \sin(2t)]\}$. Define x_1 to be that value of $x \equiv 2t - \pi/2$ such that $(1+\alpha)\zeta F - 2\varepsilon \cos x = 2(1-\alpha)\zeta$ (see Fig. 2), where F will be understood to be $F_{\min}$ for notational simplicity. Thus, defining $\mu = 2\varepsilon/\zeta$,

$$\text{(A.1)} \qquad \mu \cos x_1 = (F-2) + \alpha(F+2)$$

and

$$\text{(A.2)} \qquad I = -\int_0^{x_1} [(1-\alpha)F - \mu\cos x]\,dx - 2(1-\alpha)[\pi - x_1],$$

μ_1 will be defined by $I \equiv 0$ (i.e., since for $\mu < \mu_1$, I is negative, $\mu < \mu_1$ is the stability requirement) and it will be desired to find $\alpha \in (0, 1]$ such that μ_1 is a maximum; hence

$$\text{(A.3)} \qquad \mu_1^* \equiv \max_{0<\alpha\leq 1} \mu_1(\alpha)$$

is sought. Setting $I = 0$ yields a second equation in $\mu_1(\alpha)$ and $x_1(\alpha)$ to be used in conjunction with (A.1). Directly from (A.1) and (A.2), the system of simultaneous equations is

$$\text{(A.4)} \qquad \begin{aligned} \mu_1 \cos x_1 &= (F-2) + \alpha(F+2), \\ \mu_1(\sin x_1 - x_1\cos x_1) &= 2\pi(1-\alpha). \end{aligned}$$

It is desired to find $d\mu_1/d\alpha$ and that value of α such that $d\mu_1/d\alpha = 0$, $d^2\mu_1/d\alpha^2 < 0$. Differentiating (A.4) yields

$$\frac{dx_1}{d\alpha} = \frac{\cos x_1(d\mu_1/d\alpha) - (F+2)}{\mu_1 \sin x_1},$$

$$\frac{dx_1}{d\alpha} = \frac{-2\pi - (\sin x_1 - x_1\cos x_1)\,d\mu_1/d\alpha}{\mu_1 x_1 \sin x_1},$$

which by resubstitution of (A.4) yields

$$\text{(A.5)}\qquad \frac{d\mu_1}{d\alpha} = \frac{(F+2)x_1 - 2\pi}{\sin x_1}, \qquad \frac{dx_1}{d\alpha} = \frac{-4\pi F}{\mu_1^2 \sin^2 x_1}.$$

A second differentiation yields

$$\frac{d^2\mu_1}{d\alpha^2} = -\mu_1\left(\frac{4\pi F}{\mu_1^2 \sin^2 x_1}\right)^2 < 0.$$

Thus, as $d^2\mu_1/d\alpha^2$ is demonstrably always negative, there is only one zero of $d\mu_1/d\alpha$ (at $x_1^* \equiv 2\pi/(F+2)$, and it corresponds to the maximum of $\mu_1(\alpha)$.

It must be recalled that $\alpha^* \equiv \alpha(x_1^*)$ must lie in the range $(0, 1]$; if $\alpha^* \leqq 0$ then $\alpha = 0^+$ must be used and if $\alpha^* > 1$, then $\alpha = 1$. In order to determine those values of $F_{\min}$ for which $\alpha^* \in (0, 1]$, it is necessary to inspect

$$\text{(A.6)}\qquad \mu_1^0 = 2\frac{x_1^*}{\sin x_1^*}F,$$

and

$$\text{(A.7)}\qquad \alpha^* = \frac{2}{\pi}x_1^*(\pi - x_1^*)\cot x_1^* + \left(\frac{2x_1^*}{\pi} - 1\right),$$

which may be derived by substituting $x_1^* = 2\pi/(F+2)$ into (A.4). Clearly, for $F \in (0, \infty)$, $x_1^* \in (0, \pi)$. By inspection, $\alpha^*(0) = +1$, $a^*(\pi/2) = 0$ and $\alpha^*(\pi) = -1$. It will now be shown that $d\alpha^*/dx_1 \leqq 0$ for all $x_1^* \in (0, \pi)$. This, in addition to the values of α^* at $x_1^* = 0, \pi/2$ and π, will establish that $\alpha^* \in (0, 1]$ only for $x_1 \in (0, \pi/2)$ and for $x_1 \in [\pi/2, \pi)$, it is necessary to choose $\alpha = 0^+$.

From (A.7), it may be seen that $d\alpha^*/dx = -2h(x)/\pi \sin^2 x$, where

$$\text{(A.8)}\qquad h(x) \equiv x(\pi - x) - \sin^2 x - (\pi - 2x)\sin x \cos x.$$

It would thus suffice to show that $h(x) > 0$ for all $x \in (0, \pi)$. Note that $h(0) = 0$, $h(\pi/2) = (\pi/2)^2 - 1 > 0$ and $h(\pi) = 0$. Thus $h(x) > 0$, for all $x \in (0, \pi)$ if $dh/dx > 0$, $x \in (0, \pi/2)$ and $dh/dx < 0$, $x \in (\pi/2, \pi)(h(x)$ increases monotonically from 0 to $(\pi/2)^2 - 1$, then decreases monotonically to 0). It may be shown that $dh/dx = 2(\pi - 2x)\sin^2 x$, which satisfies the condition that is sufficient to ensure that $d\alpha^*/dx^* < 0$.

According to the previous discussion, there are thus two cases:

(a) $2 < F_{\min} < \infty$ (or $x_1^* \in (0, \pi/2)$).

$$\text{(A.9)}\qquad \mu_1^* = \mu_1^0 = \frac{4\pi F_{\min}}{(F_{\min} + 2)\sin(2\pi/(F_{\min} + 2))},$$

(b) $0 < F_{\min} \leqq 2$, (or $x_1^* \in (\pi/2, \pi)$).

For $\alpha = 0^+$, μ_1^* is the solution of the simultaneous equations

$$\text{(A.10)}\qquad \mu_1^* \cos x = (F - 2) < 0, \qquad \mu_1^*(\sin x - x\cos x) = 2\pi.$$

As x satisfying the above is in the range $(\pi/2, \pi)$, substitute $y = \pi - x$, $\cos y = -\cos x$ and $\sin y = \sin x$; therefore

$$\mu_1^* \cos y = 2 - F,$$

$$\mu_1^* (\sin y + (\pi - y) \cos y) = 2\pi.$$

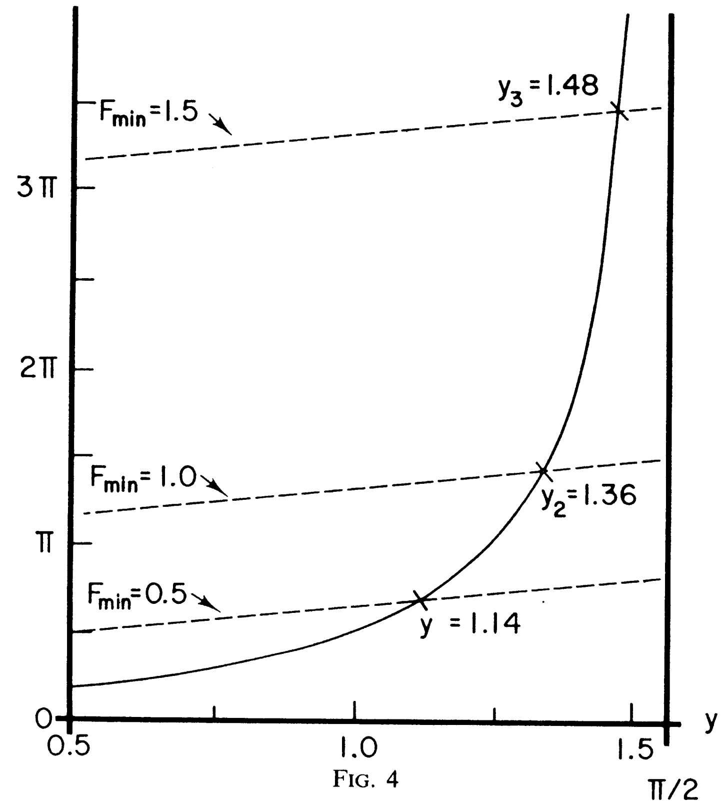

FIG. 4

TABLE 1

F_{min}	y	μ_1^*
0^+	0^+	2.0^+
0.5	1.1425	3.61
1.0	1.352	4.61
1.5	1.479	5.45
2.0^+	–	6.28
2.5	–	7.09
3.0	–	7.93
4.0	–	9.67
5.0	–	11.50

By eliminating μ_1^*, it is obvious that y must satisfy

(A.11)
$$\tan y = y + \frac{\pi F_{min}}{2 - F_{min}},$$

which is best solved by graphical techniques. By inspection, $F_{\min} = 0^+$ yields $y = 0^+$, so that $\mu_1^* = 2^+$. Values of y are found graphically for $F_{\min} = 0.5, 1.0$ and 1.5 (see Fig. 4) to obtain sample values of μ_1^* for $F_{\min} \in (0, 2)$. Values of μ_1^* are calculated for several values of $F_{\min} \in (2, \infty)$ using (A.9). Relevant data is contained in Table 1. The stability boundary $\varepsilon^{**} = \frac{1}{2}\zeta\mu_1^*$ is plotted versus $F_{\min}$ in Fig. 3, as is ε^* (4.4).

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