

# Tutorial Guide: Enhanced MATLAB Tools for Linear and Nonlinear System Stability Analysis

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*Abstract*<sup>1</sup>: Three MATLAB-based tools have been developed for the convenient assessment of stability conditions for linear and nonlinear systems. The first is a more helpful and definitive version of `nyquist`, which has features of automatic zooming (to show all crossings of the real axis and thus display the corresponding critical gain points) and which displays on the real axis a numeral ('0', '1' etc.) that represents the number of unstable (right-half-plane) poles that would result for a closed-loop system with feedback gain  $k$  if  $-1/k$  is located in that region. The second tool is built on this extended `nyquist` command, and makes the application of the nonlinear time-varying system Circle Criterion equally simple and definitive. The third routine implements the Popov criterion for nonlinear time-invariant systems, which typically produces substantially less restrictive sector constraints than the Circle Criterion, as one would expect due to assumption of time invariance. Examples are presented to show the use and efficacy of these extensions within the MATLAB environment.

**Keywords:** Stability; linear systems; nonlinear systems; time-varying systems; Nyquist Criterion; Circle Criterion; Popov Criterion.

## 1 Introduction

Existing MATLAB commands for the stability analysis of linear systems are not easy to interpret by the less-than-expert user, and tools for assessing the stability of nonlinear systems are nonexistent in MATLAB. In the linear case, interpreting the usual MATLAB Nyquist plot is not hard if the plant is stable and the  $G(j\omega)$  locus is not complicated, but it may be confusing, especially if there are multiple real-axis crossings and/or open-loop poles on the imaginary axis or in the right-half-plane (RHP). In the nonlinear case, none of the classical absolute stability criteria are implemented in MATLAB, so we have addressed that lack by creating comparably user-friendly tools

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<sup>1</sup>This Tutorial Guide is based on two papers, "Enhanced MATLAB Tools for Linear and Nonlinear System Stability Analysis", which appeared in *Proc. IFAC CACSD'97*, Gent, Belgium, 28-30 April 1997; and "MATLAB Tools for Linear and Nonlinear System Stability Theorem Implementation", from *Proc. IEEE Conference on Control Applications*, Hartford, Connecticut, October 1997, both co-authored by J. H. Taylor and C. Chan.

for the application of the Circle Criterion for the nonlinear time-varying case and the Popov Criterion for time-invariant nonlinear systems [1, 2, 3].

## 2 Stability Criteria

In this section we state the linear and nonlinear stability criteria, emphasizing their classical geometric interpretations. As a preliminary, the class of systems considered is depicted in Fig. 1, where it is assumed that the forward-path linear dynamic system is expressed in state-space form as:

$$G(s) = C(sI - A)^{-1}B + D \quad (1)$$

or in Laplace notation as a ratio of polynomials, with the order of the numerator not exceeding that of the denominator:

$$G(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}, \quad m \leq n \quad (2)$$

and that the open loop is completed by a linear gain  $k$  or time-invariant nonlinearity  $f(\cdot)$  or nonlinear time-varying operator  $f(\cdot, t)$ .

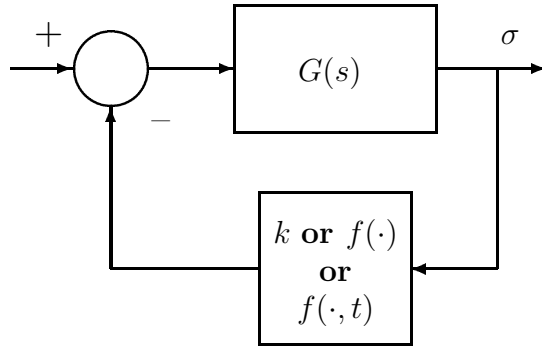


Figure 1: Closed-loop Linear/Nonlinear System

The Nyquist criterion states that the closed-loop system in Fig. 1 with  $k$  in the feedback path will be stable if the point  $-1/k$  is not in the  $G(s)$ -map of the right half of the  $s$ -plane (RHP). In the case of a stable plant  $G(s)$ , this map is simply that region to the right as one traverses the Nyquist plot of  $G(j\omega)$ . In the case of a plant with  $q$  poles in the right-half-plane, this map must take into consideration  $q$  map layers due to excluding these singularities by traversing a small circle around each in the counter-clockwise sense, which results in a large circle traversed in the clockwise sense for each. As an example, consider the unstable plant:

$$G(s) = \frac{s + 2}{s^2 - 4s - 5} = \frac{s + 2}{(s + 1)(s - 5)} \quad (3)$$

Then the  $s$ -plane region being mapped and the resulting complete Nyquist diagram are portrayed in Fig. 2. Observe that the only range of  $k$  leading to stability is  $4 < k < \infty$ ; the corresponding region for  $-1/k$  is shown with the numeral 0 on the real axis to denote no closed-loop poles in the RHP. For  $2.5 < k < 4$  there will be 2

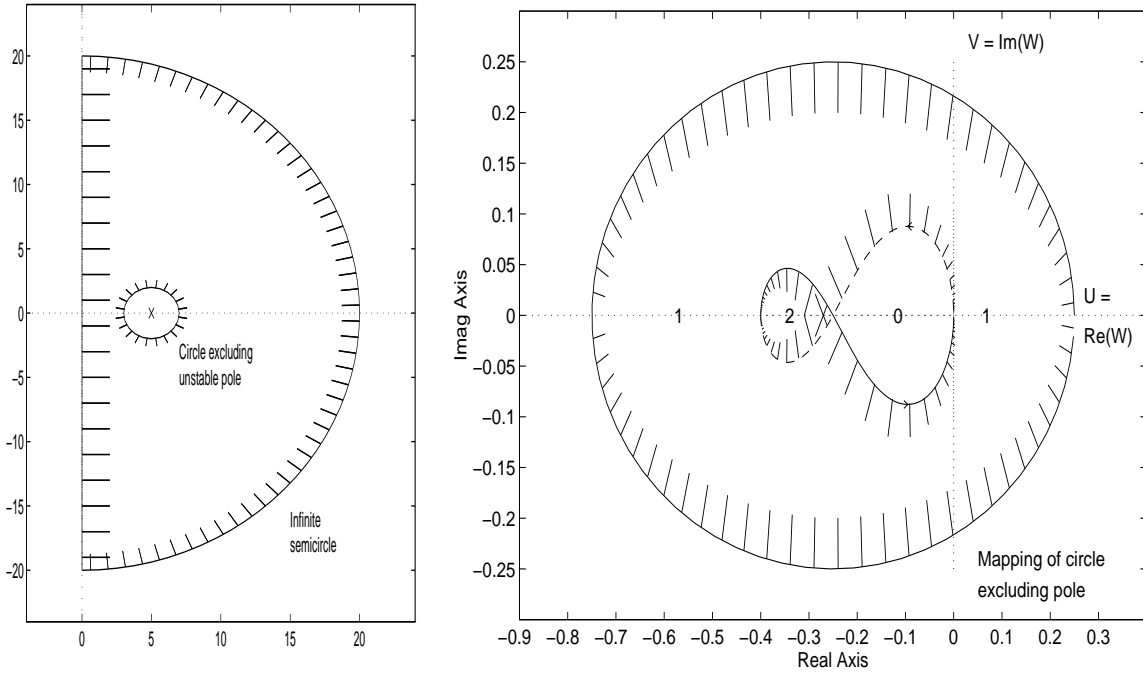


Figure 2:  $s$ -plane region mapped and  $G(s)$ -map for the Nyquist criterion

unstable closed-loop poles, and for the range  $-\infty < k < 2.5$  there will be one; again, the numerals on the real axis indicate these results.

The Circle Criterion [1, 3] states that the closed-loop system in Fig. 1 with  $f(\cdot, t)$  in the sector  $[\underline{F}, \overline{F}]$ , i.e.,

$$\underline{F} \leq \frac{f(\sigma, t)}{\sigma} \leq \overline{F}, \quad \forall \sigma, t \quad (4)$$

is *absolutely stable* (uniformly asymptotically stable in the large – UASIL – for any nonlinearity satisfying Eqn. 4) if one can draw a circle on the  $G = U + jV$  plane whose diameter is defined by the points  $V = 0, -1/\underline{F} < U < -1/\overline{F}$  and whose interior has no points in common with the  $G(s)$ -map of the right half of the  $s$ -plane. In essence, this circle must lie in a region of the Nyquist plane where there are zero mapping layers in the sense of the Nyquist criterion stated above. The case  $\underline{F} < 0 < \overline{F}$  is interesting in that the “diameter” so defined passes through the point  $\infty$  and the “inside” of the circle is actually what would ordinarily be considered the outside (exterior points), as shall be illustrated in the examples of Section 5.

Finally, the Popov Criterion [2, 3] states that the closed-loop system in Fig. 1 with  $f(\cdot)$  in the sector  $[0, \overline{F}]$ , i.e.,

$$0 \leq \frac{f(\sigma)}{\sigma} \leq \overline{F}, \quad \forall \sigma \quad (5)$$

is *absolutely stable* (UASIL under the sector constraint in Eqn. 5) if one can draw a line on the *Popov plot*, i.e., polar plot of  $\widehat{G} = U + j\widehat{V} \triangleq U + j\omega V$ , passing through the point  $-1/\overline{F}$  such that the Popov locus of  $G(j\omega)$  lies entirely to the right of the line.

Unfortunately, some of the elegance of the Circle Criterion is lost, due to this use of a *modified* frequency-domain plot. In particular, the criterion for the general finite-

sector case  $[\underline{F}, \overline{F}]$  cannot be stated in graphical terms; rather, one must resort to the **finite-sector transform** [3] to convert the case  $[\underline{F}, \overline{F}]$  to  $[0, (\overline{F} - \underline{F})]$  by applying the above criterion to the transformed system  $\tilde{G}, \tilde{f}$ ,

$$\tilde{G} \triangleq \frac{G}{1 + \underline{F}G}, \quad \tilde{f} \triangleq f(\sigma) - \underline{F}\sigma \quad (6)$$

which we do automatically in our tool<sup>2</sup>.

### 3 Description of MATLAB Stability Routines

Given the above problem definition, the following “help” displays constitute a concise “user’s guide” for the routines `newnyq`, `circle` and `popov`:

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`NEWNYQ` A new & improved version of Nyquist frequency response for SISO continuous-time linear systems (see NYQUIST). `NEWNYQ(A,B,C,D)` produces a Nyquist plot. The frequency range, number of points and scaling are set automatically, and a "zoom" feature displays all real-axis crossings. In addition, the `k` range(s) for closed-loop stability are reported, and the number of unstable closed-loop poles for  $-1/k$  on various regions of the real axis are displayed.

`NEWNYQ(A,B,C,D,1)` will cause the right-half-plane mapping of  $G(s) = C(sI-A) B + D$  to be indicated by hatching.

`[KMIN,KMAX] = NEWNYQ(A,B,C,D)` will return the minimum and maximum values of `K` for stability in addition to plotting.

`[KMIN,KMAX,RE,IM,W] = NEWNYQ(A,B,C,D)` will return the real and imaginary parts of  $G(j\omega)$  in addition, with no plot.

`NEWNYQ(NUM,DEN)` and `NEWNYQ(NUM,DEN,1)` are corresponding variants for  $G(s)$  provided in ratio of polynomial form.

C. Chan and J.H. Taylor, Univ. of New Brunswick, Feb. 1997.

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`CIRCLE` An implementation of the Circle Criterion for nonlinear feedback control system with a linear part  $G(s)$  and a nonlinearity lying in a sector  $[F_{\min}, F_{\max}]$ , i.e.,  

$$F_{\min} < f(y,t)/y < F_{\max}$$
The nonlinear system is uniformly asymptotically stable in the large (UASIL) if the circle whose diameter is defined by the points  $-1/F_{\max} < \text{Re}(G(j\omega)) < -1/F_{\min}$ ,  $\text{Im}(G(j\omega)) = 0$  contains no interior points of the right-half-plane (RHP) mapping of  $G(s)$ .

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<sup>2</sup>While there is a Parabola Criterion for the general finite-sector case [3], it is unattractive, since it may be more strict than the Popov Criterion applied using the finite-sector transform.

CIRCLE(A,B,C,D,Fmin,Fmax) will check if the CC is satisfied for the sector-bounded nonlinearity and  $G(s) = C(sI-A) B + D$

CIRCLE(A,B,C,D,Fmin) will determine the maximum value of  $F_{max}$  for which the CC is satisfied.

CIRCLE(A,B,C,D,NaN,Fmax) will determine the minimum value of  $F_{min}$  for which the CC is satisfied.

[FMIN,FMAX] = CIRCLE(A,B,C,D,FMIN) will return the sector limits  $F_{min}$  &  $F_{max}$  for stability in addition to plotting.

CIRCLE(NUM,DEN,Fmin,Fmax) et cetera are corresponding forms for  $G(s)$  provided in ratio of polynomial form.

This routine is built on NEWNYQ, a new & improved version of the Nyquist criterion with advanced graphics and reporting.

C. Chan and J.H. Taylor, Univ. of New Brunswick, Feb. 1997.

Reference: Narendra & Taylor "Frequency Domain Criteria for Absolute Criteria", Academic Press, 1973.

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POPOV

An implementation of the Popov Criterion (PC) for a non-linear feedback control system with a linear part  $G(s)$  and a time-invariant nonlinearity  $f(y)$  lying in a sector  $[ F_{min}, F_{max} ]$ , i.e.,

$$F_{min} < f(y)/y < F_{max}$$

If  $F_{min}$  is not 0,  $G(s)$  and  $f(y)$  are transformed to obtain the special case  $[ 0, (F_{max} - F_{min}) ]$  by defining  $g(y) = f(y) - F_{min}y$ ,  $G(s) = G(s)/(1 + F_{min}G(s))$ . The special case is uniformly asymptotically stable in the large (UASIL) if a line can be drawn through the point  $-1/(F_{max} - F_{min})$  such that the Popov locus of  $G(jw)$  lies to the right; the Popov locus is the plot of  $w \cdot \text{Im}(G(jw))$  vs  $\text{Re}(G(jw))$ .

POPOV(A,B,C,D,F\_min,F\_max) will check if the PC is satisfied for the sector-bounded nonlinearity and  $G(s) = C(sI-A) B + D$

POPOV(A,B,C,D,F\_min) will determine the maximum value of  $F_{max}$  for which the PC is satisfied.

POPOV(A,B,C,D,NaN,F\_max) will determine the minimum value of  $F_{min}$  for which the PC is satisfied.

POPOV(NUM,DEN,F\_min,F\_max) et cetera are corresponding forms for  $G(s)$  provided in ratio of polynomial form.

[FMIN,FMAX]=POPOV(A,B,C,D,F\_min) et cetera will return the

sector bounds for which the PC is satisfied (even if, for example, F\_min, F\_max are supplied for which the PC is NOT OK).

C. Chan and J.H. Taylor, Univ. of New Brunswick, July. 1997.

Reference: Narendra & Taylor "Frequency Domain Criteria for Absolute Criteria", Academic Press, 1973.

## 4 Algorithm Overview

The Circle Criterion is implemented as follows: First, the finite-sector transform (Eqn. 6) is applied to reduce the criterion for UASIL to requiring that the Nyquist locus of  $\tilde{G}(j\omega)$  must lie to the right of a vertical line passing through  $\tilde{U} = -1/(\overline{F} - \underline{F})$ . (With a zero sector bound, the circle degenerates into a vertical line.) Then we analytically solve the polynomial equation corresponding to  $d\tilde{U}(j\omega)/d\omega = 0$  using the `root` command, and thus locate all local minima and maxima of  $\tilde{U}(j\omega) \triangleq \text{Re}\tilde{G}(j\omega)$ . These are inspected to determine the global minimum, i.e., the left-most point on the Nyquist locus of  $\tilde{G}(j\omega)$ . Denoting the real part of this left-most point as  $R_{min}$ , the maximum upper sector bound permitted by the Circle Criterion is given by  $\overline{F} = \underline{F} - 1/R_{min}$ . This maximum upper bound is reported to the user if only the lower bound is given (`circle(num,den,Fmin)`), or it is used to test the user's upper bound if it is also provided (`circle(num,den,Fmin,Fmax)`).

The algorithm for the Popov Criterion is also based on the analysis of  $\tilde{G}(j\omega)$ . It is more complicated, since the Popov line may have an arbitrary orientation (angle with respect to the real axis). In essence, we define an auxiliary function  $F(\theta)$  to be the difference between the two left-most points of the Popov locus rotated by the angle  $\theta$  and use the MATLAB function-solver `fzero` to find  $\theta^*$  such that this difference is zero – this is the slope of the Popov line. Once the slope is known, the point where the Popov line crosses the real axis is readily determined; denoting this point as  $R_{min}$ , the maximum upper sector bound permitted by the Popov Criterion is again given by  $\overline{F} = \underline{F} - 1/R_{min}$ . This upper bound is reported to the user if only the lower bound is given (`popov(num,den,Fmin)`), or it is used to test the user's upper bound if it is also provided (`popov(num,den,Fmin,Fmax)`).

## 5 Examples

First we illustrate the use of `newnyq` and `circle` on a relatively simple stable plant:

$$G(s) = \frac{s + 1}{s^4 + 2s^3 + 25s^2 + 3s + 1} \quad (7)$$

Referring to Fig. 3, we note that the upper plot is similar to that produced by the MATLAB `nyquist` command except for the numerals on the real axis indicating the corresponding number of unstable closed-loop poles. The lower plot in Fig. 3 illustrates the feature of automatic zooming (to show all crossings of the real axis);

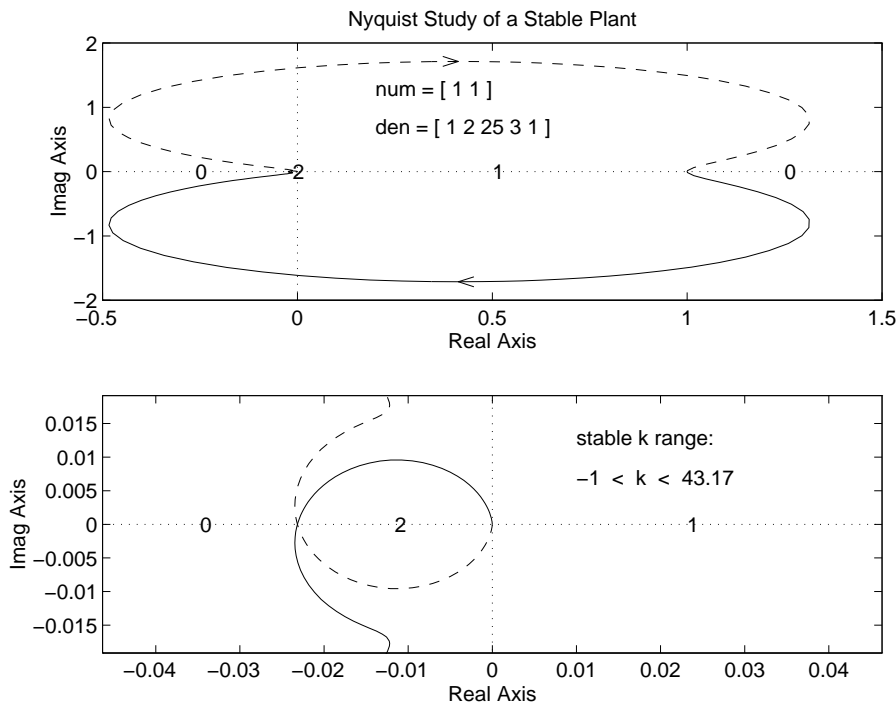


Figure 3: Nyquist Criterion Example (Stable Plant, with Zooming)

the only manual operations in producing this figure were the `text` commands to document the example (record `num` and `den` and the stable  $k$  range). The invocation and report that `newnyq` provides for this example are:

```
>> newnyq(num,den)

stable k range -1 < k < 43.17
```

Finally, the numerals 0, 1, 2 written on the real axis represent the number of right-half-plane closed-loop poles that would result if  $-1/k$  were to lie in each region.

Once the Nyquist plot has been viewed, one may request that the CC be applied to the same  $G(j\omega)$  for any sector lower bound in the Nyquist range, for example,  $F_{min} = 2.5$ . The CC locus is shown on the Nyquist plot (Fig. 4), and the report that `circle` provides after it is invoked for this example is:

```
>> circle(num,den,2.5)

circle criterion is satisfied
maximum sector bound F_max = 11.5925
```

Again – the only manual steps in preparing Fig. 4 were the documentation `num =` , `den =` , `F_min =` etc. defining the example and the result. The interpretation of this result is that  $f(\cdot, t)$  in Fig. 1 must lie in the sector  $[ 2.5 , 11.59 ]$ , i.e.,

$$2.5 \leq \frac{f(\sigma, t)}{\sigma} \leq 11.59 , \quad \forall \sigma, t \quad (8)$$

in order to guarantee absolute stability.

We conclude this example by illustrating the observation in Section 2 that the “interior” of the CC circle may include the point  $\infty$  if  $F_{min} < 0$  by taking  $F_{min} = -0.5$  for

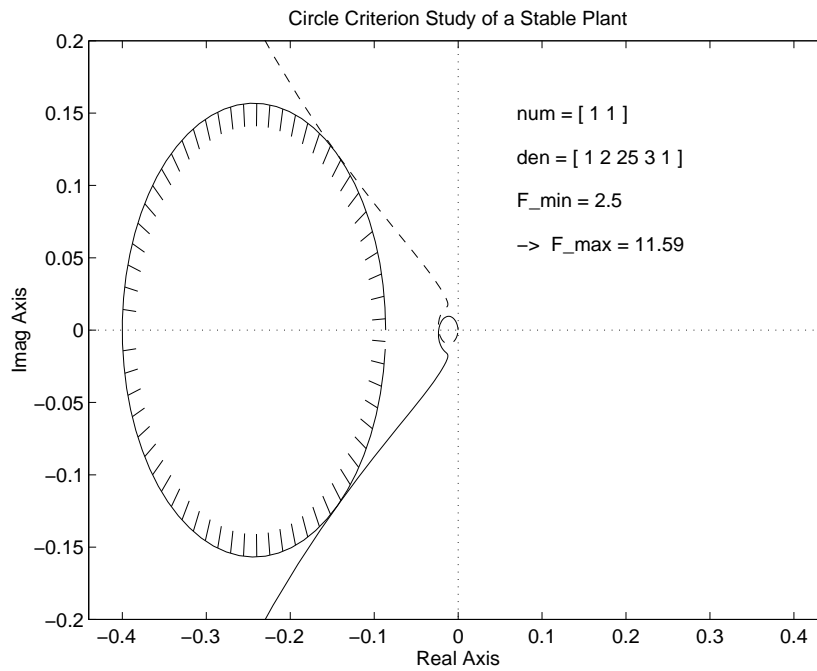


Figure 4: Circle Criterion Example (Stable Plant)

the above plant. The result is shown in the following report and Fig. 5:

```
>> circle(num,den,-0.5)
```

```
circle criterion is satisfied
maximum sector bound F_max = 0.68971
```

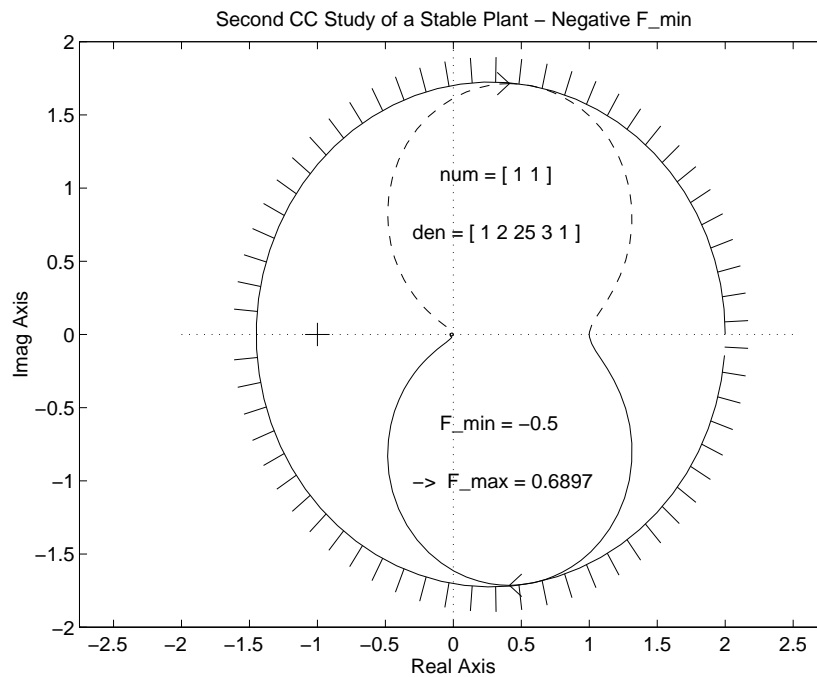


Figure 5: Circle Criterion Result for Negative  $F_{\min}$

Next, we provide the MATLAB plots obtained applying `newnyq` and `circle` to the



unstable plant in Eqn. 3. The resulting “new Nyquist” plot is depicted in Fig. 6;

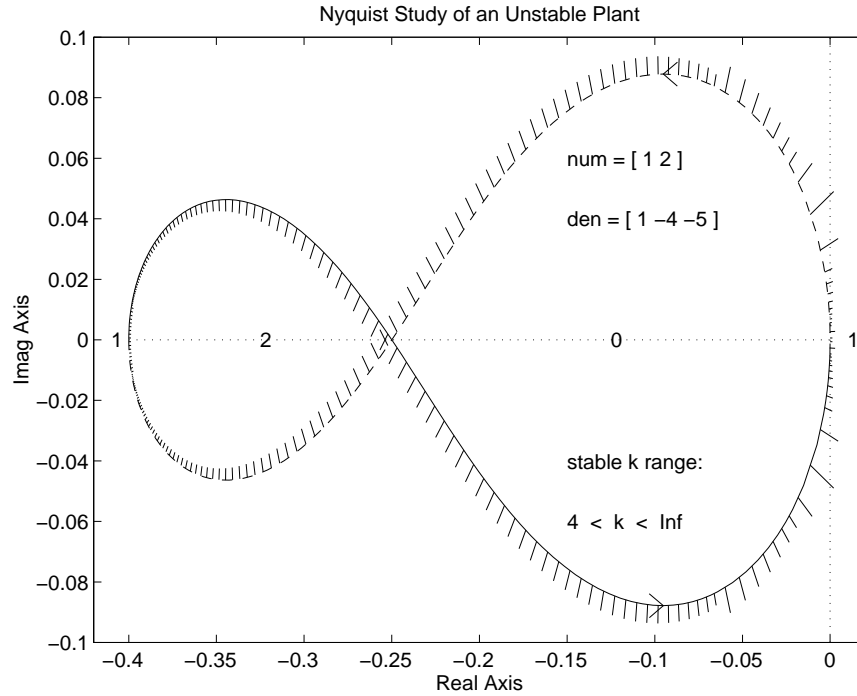


Figure 6: Nyquist Criterion Example (Unstable Plant)

note that it is similar to Fig. 3 except that the “large (infinite) circle” arising from the exclusion of the right-half-plane singularity is not shown (to avoid unnecessary clutter); that circle is, however, the basis for the numeral 1 at the far left on the real axis (meaning that any  $k$  such that  $-1/k < -0.4$  will result in a closed-loop system with one unstable pole). The region  $-0.25 < -1/k < 0$  on the real axis has the numeral 0, because the points are to the *left* of the  $G(j\omega)$  locus meaning that points there are no longer in the right-half-plane map of  $G(s)$ . In light of this, the report that `newnyq` provides in this case is:

```
>> newnyq(num,den)

stable k range
4 < k < Inf
```

To illustrate the application of the Circle Criterion tool to the same unstable plant, we specify a lower bound of  $\underline{F} = 5$ , which is in the Nyquist range as required, and the report that `circle` provides for this example is:

```
>> circle(num,den,5)

circle criterion is satisfied
maximum sector bound F_max = 27.83
```

Again, we note that the corresponding circle in the Nyquist plane (Fig. 7) lies completely within the “no mapping” portion of the plot.

Another feature of `circle` allows us to specify the upper bound ( $F_{max}$ ) and obtain the minimum corresponding lower bound that guarantees UASIL. To demonstrate this we issue the command `circle(num,den,NaN,100)` on the above unstable plant. We

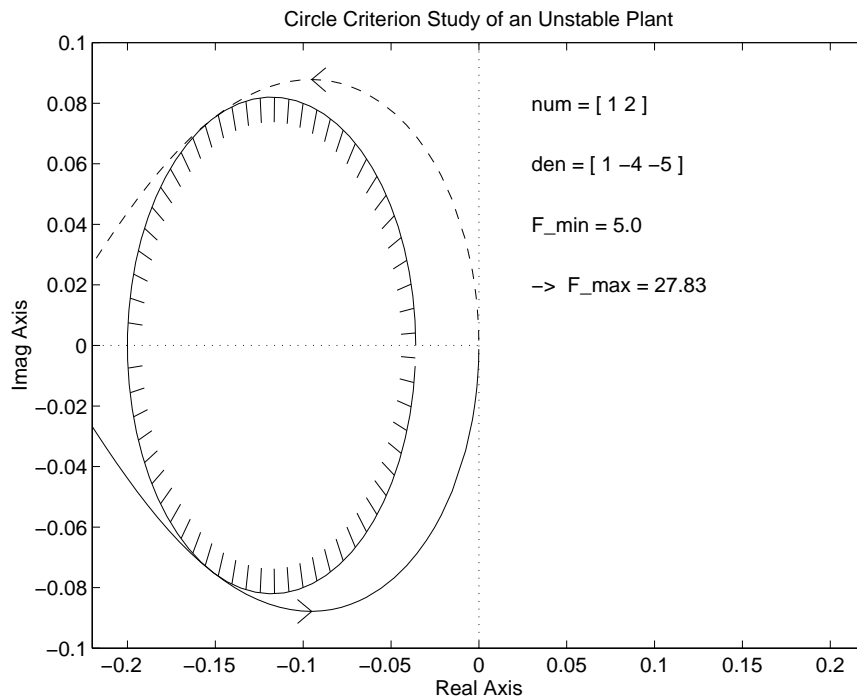


Figure 7: Circle Criterion Example (Unstable Plant)

receive the following report and plot:

```
>> circle(num,den,NaN,100)
```

```
circle criterion is satisfied  

minimum sector bound F_min = 5.392
```

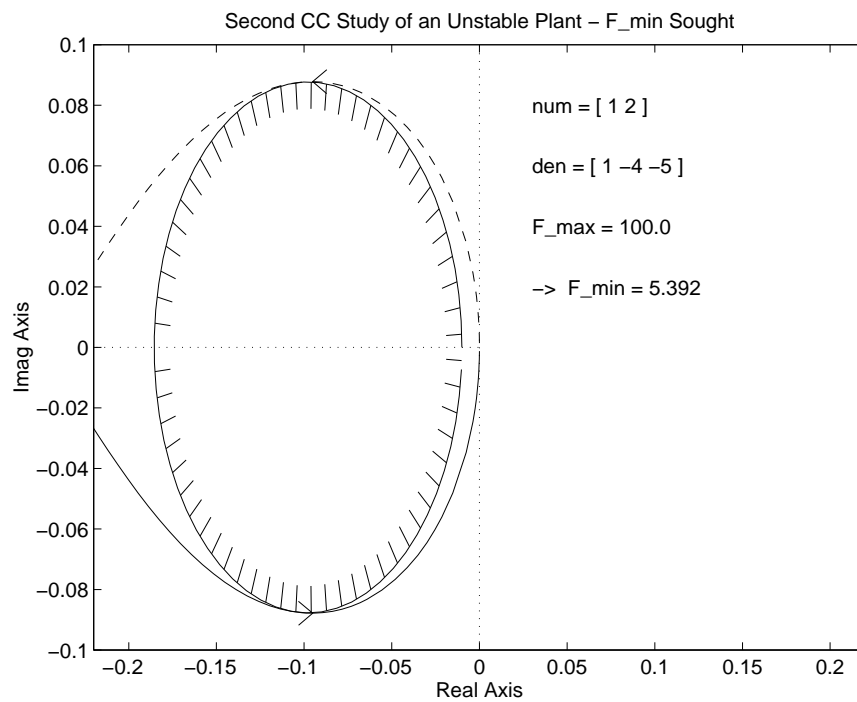


Figure 8: Circle Criterion Example (Upper Bound Provided)

Finally, we apply the Popov Criterion tool to the same stable plant  $G(s)$  as before (Eqn. 7), first for the case when  $\underline{F} = 0$ , so the Popov locus is rendered for the same  $G(j\omega)$  as shown in Fig. 3: The report for this case is as follows:

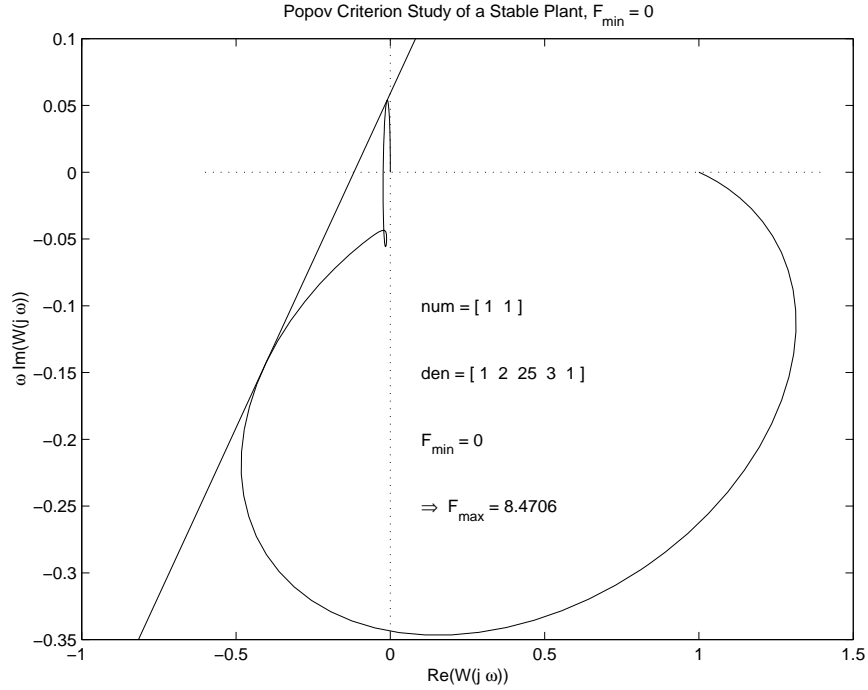


Figure 9: Popov Criterion Example (Stable Plant,  $\underline{F} = 0$ )

```
>> popov(num,den,0)
```

```
popov criterion is satisfied
maximum sector bound F_max = 8.4706
```

Now, the same tool is applied to the case  $\underline{F} = 2.5$ , which may be compared directly with the CC example depicted in Fig. 4: The Popov plot of the transformed system  $\tilde{G}$  is depicted in Fig. 10 and the report for this case is as follows:

```
>> popov(num,den,2.5)
popov criterion is satisfied
```

```
maximum sector bound F_max = 25.019
```

The CC gave the substantially smaller upper sector bound  $F_{\max} = 11.59$ , which is not surprising considering that the CC guarantees UASIL for nonlinear *time-varying* systems while the Popov Criterion requires that  $f$  be time invariant. In fact, the CC upper sector bound may be obtained by drawing a *vertical* line that just touches the Popov locus in Fig. 10 to determine  $(\overline{F} - \underline{F})$ , since the left-most extent of the Popov and Nyquist loci are the same (the abscissas for these plots are the same,  $U = \text{Re}[G(j\omega)]$ ).

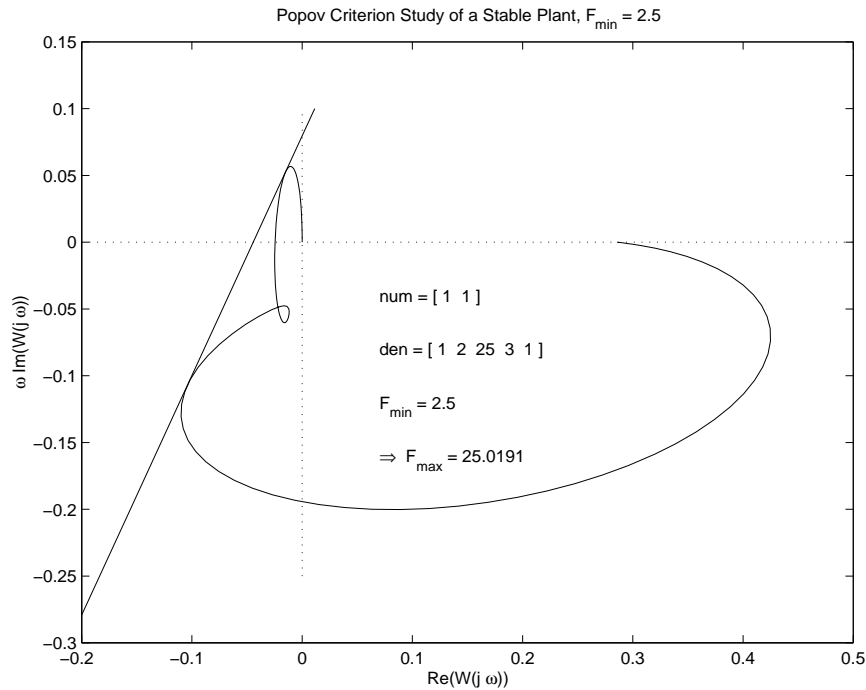


Figure 10: Popov Criterion Example (Stable Plant,  $\underline{F} = 2.5$ )

## 6 Conclusion

The routines presented above provide a simple environment for the determination of stability conditions for closed-loop systems with linear plants in the forward path and linear or nonlinear operators in the feedback path (Fig. 1). In fact, the nonlinear operator may occur at any point in the loop; Fig. 1 merely represents a convenient canonical form. We trust that the convenience and added support these tools supply can be readily appreciated from the examples presented in Section 5.

We have installed these routines on our web page, URL = <http://www.ee.unb.ca/jtaylor/> for your convenient access. A copy of this “User’s Guide” is also available there, along with two matlab scripts that demonstrate the application of these tools to a wider variety of applications (`run_criteria.m` and `run_popov.m`).

## 7 References

- [1] Narendra, K. S. and Goldwyn, R. M. “A Geometrical Criterion for the Stability of Certain Nonlinear Nonautonomous Systems”, *IEEE Trans. Circuit Theory, CT-11*, No. 3, 1964.
- [2] Popov, V. M., “Nouveaux Criteriums de Stabilité pour les Systemès Automatiques Non-Linéaires”, *Revue d’Electrotechnique et d’Energetique, Acad. de la Populaire Romaine*, 5, No. 1, 1960.
- [3] Narendra, K. S. and Taylor, J. H. *Frequency Domain Criteria for Absolute Stability*, Academic Press Inc., Reading, MA, 1973.