Modeling a Quad Rotor UAV

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Common Assumptions for Quad Rotors

1. The earth is flat, stationary and therefore an approximate inertial reference frame.
2. The atmosphere is at rest relative to the ground (zero wind)
3. The motors response rate is fast enough to neglect
4. The flapping angle of the rotors is negligible
5. Forces are symmetric in flight and act at the center of gravity.
6. Change in relative air velocity is negligible

State Formulation

Used in AUV's and UAV's the following state space representation is very convent

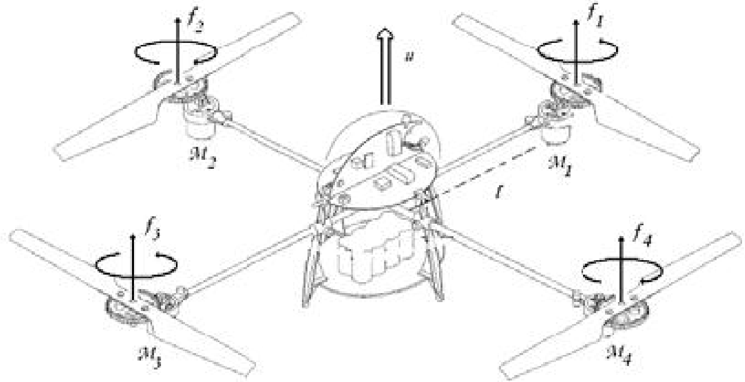
The UAV's position is expresses relative to an inertial reference.

$$\eta_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \eta_2 = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \quad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{bmatrix}$$

Also the ν vector holds the linear and angular velocity

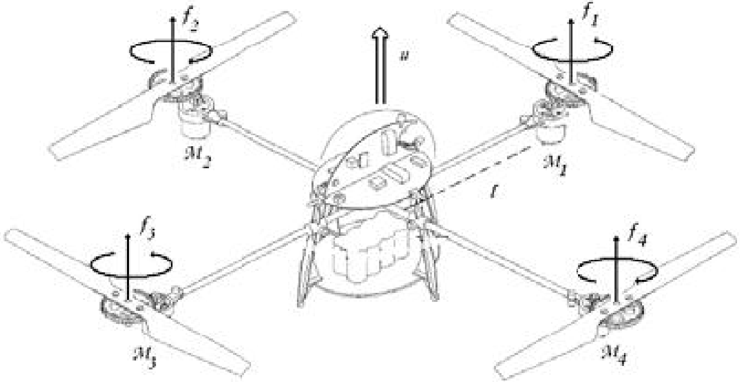
$$\nu_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \nu_2 = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad \nu = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix}$$

Quad Rotors Torque Cancellation



In order to have zero resultant torque on the quad two props must spin in opposition to the others. A less elegant way is to change the angle of two opposing motors to if counter rotating props are not available.

Quad Rotors Control Inputs



Control inputs are $\omega_1, \omega_2, \omega_3, \omega_4$

These settings can be treated as a linear function of the maximum

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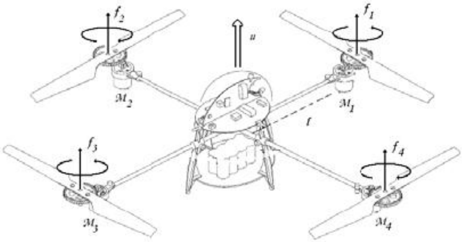
Each rotor produces a force on the system, a torque and a moment



The Lagrangian method in a nut shell

The Euler-Lagrangian method was developed in 1750 by Euler and Lagrange in mathematical formulating the calculus of variation as Euler coined it.

The basic method to find the equation of motion for a give system

- Calculate the Lagrangian L
  $L = kineticenergy - potentialenergy$
- Compute $\frac{\delta}{\delta q}$ where $q = mv$
- Compute $\frac{\delta}{\delta \dot{q}}$ from $\frac{d}{dt}\frac{\delta}{\delta \dot{q}}$ where $\dot{q}$ is treated as a variable in and of its self
- This leads the the Euler-Lagrangian equation by equating $\frac{\delta}{\delta q} = \frac{d}{dt}\frac{\delta}{\delta \dot{q}}$

Applying the Euler-Lagrangian method to determine the math model

Using the definitions for position and orientation the Translation kinetic energy for the system can be expressed as

$T_{trans} \equiv 1/2mV^2 \equiv \frac{m}{2}\dot{\eta_1}^t\dot{\eta_1}$

where m is the mass of the craft

In a similar way the rotational kinetic energy

$T_{rot} \equiv \frac{1}{2}\dot{\eta_2}^t J\dot{\eta_2}$

where J is the inertia matrix

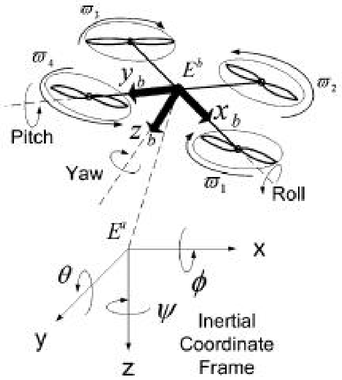
The only potential energy considered is gravitational in nature

$U = mgz$

The Quad Rotor Model

The multiple forces and moments can be simplified into a single force and three moments

The generalized force can be calculated using
$u = f_1 + f_2 + f_3 + f_4$
where
$f_i = k_i \omega_i, i = 1, ..., 4$ and k_i is a constant and ω_i is the motor angular speed



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The generalized moments

$$\tau \equiv \begin{bmatrix} \tau_\psi \\ \tau_\theta \\ \tau_\phi \end{bmatrix}$$

where $\tau_\psi = \sum_{i=1}^{4} \tau_i$

$\tau_\theta = (f_2 - f_4)\ell$
$\tau_\phi = (f_3 - f_1)\ell$

Applying the force appropriately

The force acts normal to the body frame but can be transformed into the inertial frame by applying the rotation yaw roll pitch to F

$$F_b = \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix}$$

$F_i = RF_b$
where $R = R_\theta R_\phi R_\psi$

$$\begin{bmatrix} \cos\psi\cos\theta & \sin\psi\sin\phi & -\sin\theta \\ \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \sin\psi\sin\theta\sin\phi + \cos\theta\cos\phi & \cos\psi\sin\phi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \cos\theta\cos\phi \end{bmatrix}$$

Applying f=ma

$$m\ddot{\eta}_1 + \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = F_i$$

$$J\ddot{\eta}_2 + \dot{J}\dot{\eta}_2 - 1/2\frac{\delta}{\delta\eta_2}(\dot{\eta}^T J\dot{\eta}_2) = \tau$$

Defining the Coriolis/Centripetal Vector as
$\bar{V}(\eta_2, \dot{\eta}_2) = \dot{J}\dot{\eta}_2 - 1/2\frac{\delta}{\delta\eta_2}(\dot{\eta}^T J\dot{\eta}_2)$

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The resulting equation for τ becomes
$J\ddot{\eta_2} + \bar{V}(\eta_2, \dot{\eta_2}) = \tau$

Simplifying the Coriolis/Centripetal Vector
$\bar{V}(\eta_2, \dot{\eta_2}) = (\dot{J} - 1/2\frac{\delta}{\delta\eta_2}(\dot{\eta}^T J)\dot{\eta_2}$
$= C(\eta_2, \dot{\eta_2})\dot{\eta_2}$

Resulting Control Equations

$$m\ddot{\eta}_1 = u \begin{bmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}$$
$$J\eta_2 = -C(\eta_2, \dot{\eta}_2)\dot{\eta}_2 + \tau$$

simplifying again by changing the input variables
$\tau = C(\eta_2, \dot{\eta}_2)\dot{\eta}_2 + J\tilde{\tau}$
where $\tilde{\tau}$
$$\tilde{\tau} \equiv \begin{bmatrix} \tilde{\tau_\psi} \\ \tilde{\tau_\theta} \\ \tilde{\tau_\phi} \end{bmatrix}$$

then $\ddot{\eta}_2 = \tilde{\tau}$

Aggressive Quad Rotor Control

Aggressive Quad rotor control (video)

Based on controlling the Quad through a path and a specified velocity the quad can be given sufficient momentum to carry itself through a maneuver of specified orientation. The stabilization algorithm can then be reinitialize afterwards.