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STABILITY REGIONS FOR THE DAMPED MATHIEU EQUATION*

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1. Introduction. The stability boundaries of the undamped Mathieu equation, $\ddot{x} + (a - 2q\cos 2t)x = 0$, in the parameter space (a, q) are well known [1], [2]. The addition of a small positive damping term, $2\zeta\dot{x}$, effectively increases the area of stability in a manner that is quite difficult to ascertain exactly using classical techniques.

The first critical region in the undamped case occurs at $a = 1$; the effect of varying the coefficient of x at twice the natural frequency of the time-invariant ($q = 0$) portion of the equation is to produce instability for any $q \neq 0$. As this is not the case for $\zeta \neq 0$, the first concern of this paper will be to determine the stability boundary $q(a, \zeta)$ in the vicinity of $a = 1$ by the perturbation method. This will serve two purposes: the exact boundary at $a = 1$ will provide the basis of comparison of the results obtained using Lyapunov's direct method, and this analysis will demonstrate the difficulty of establishing a large area of stability by classical techniques.

In the main part of this paper, three stability theorems are presented and applied to the damped Mathieu equation. Again the purpose of this analysis is twofold: first to establish the stability of the solutions of the damped Mathieu equation over a sector in the a, q-plane, and to provide a realistic basis for the comparison of the efficacy of each theorem.

Finally it is the authors' intent to demonstrate the power of these quite useful and general methods for the stability analysis of linear time-varying systems of the form defined in § 2. Furthermore it should be pointed out that it is not difficult to generalize the theorems stated herein so that they can deal with the class of nonlinear time-varying systems,

$$p(D)x + k(t)f[q(D)x] = 0,$$

where $f(\cdot)$ is a continuous first and third quadrant nonlinearity, i.e., $wf(w) > 0$ for $w \neq 0$, $f(0) = 0$ (see [3]–[6], [8]).

2. System representation. In the theorems that follow, a linear system with a single time-varying parameter will be assumed to be described by the differential equation

$$[p(D) + k(t)q(D)]x = 0, \tag{2.1}$$

where $p(D) = D^n + a_n D^{n-1} + \cdots + a_2 D + a_1$, $q(D) = h_n D^{n-1} + \cdots + h_2 D + h_1$, and D^m represents the operator d^m/dt^m. The polynomials p and q are assumed to

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have no factors in common, and all roots of $p(D) = 0$ have negative real parts. It is assumed that $0 < k(t) < \bar{K}$ for all t and that $k(t)$ is absolutely continuous.

A useful alternate representation separates the system into a time-invariant plant, the input of which is related to its output through the time-varying gain $k(t)$:

$$G(s) = \frac{q(s)}{p(s)} = \frac{\mathscr{L}[\sigma(t)]}{\mathscr{L}[e(t)]}, \qquad e(t) = -k(t)\sigma(t); \tag{2.2}$$

s is the Laplace transform variable.

In the notation of (2.1), the damped Mathieu equation is

$$\ddot{x} + 2\zeta\dot{x} + [a - 2q\cos 2t]x = 0, \tag{2.3}$$

or

$$G(s) = \frac{1}{s^2 + 2\zeta s + \delta}, \qquad k(t) = (a - \delta) - 2q\cos 2t \tag{2.4}$$

in the latter notation, where $0 < \delta < a - 2q$ to ensure that $k(t) > 0$ for all t and that the roots of $p(D) = 0$ have negative real parts ($G(s)$ is a stable transfer function). It will be assumed that ζ is small ($\zeta \ll 1$) and that $a \geqq O(1) \gg \zeta$.

3. Stability theorems. The following theorems provide sufficient conditions for the stability of systems governed by (2.1) or (2.2) in the notation of the latter formulation.

THEOREM 1. (see [3], [4]). *The system as defined in* § 2 *is asymptotically stable if the Nyquist plot of* $G(j\omega)$ *lies entirely to the right of a vertical line through the point* $(-1/\bar{K}, 0)$.

THEOREM 2 (see [5]). *The system as defined in* § 2 *is asymptotically stable if a* $\lambda > 0$ *exists such that* $[1/\bar{K} + G(s)] \cdot (s + \lambda)$ *is strictly positive real* ($\mathrm{Re}\,[1/\bar{K} + G(j\omega)](j\omega + \lambda) > 0$ *for all real* ω) *and if*

$$\frac{dk}{dt} \leqq 2\lambda k\left(1 - \frac{k}{\bar{K}}\right) \quad \textit{for all } t. \tag{3.1}$$

THEOREM 3 (see [8]). *The system as defined in* § 2 *is asymptotically stable if* $G(s - \rho)$ *is a stable transfer function, a* $\lambda > 0$ *exists such that* $[G(s - \rho) + 1/\bar{K}] \cdot (s + \lambda - \rho)$ *is strictly positive real and*

$$g(t) \equiv \sup_{t} \left\{-2\rho; \left[-2\lambda\left(1 - \frac{k}{\bar{K}}\right) + \frac{1}{k}\frac{dk}{dt}\right]\right\} \tag{3.2}$$

satisfies

$$\int^{\infty} g(t)\,dt = -\infty \tag{3.3a}$$

or

$$\int_0^{T} g(t)\,dt < 0, \tag{3.3b}$$

where (3.3a) *is to be used if* $k(t)$ *is aperiodic, and* (3.3b) *is to be used if* $k(t + T) = k(t)$ *for all* t.

No proofs will be given here, as it is not the intention of the authors to develop new theory in this paper. The following comments are included only to provide historical insight, and not as a rigorous basis for the theorems.

Theorem 1 was developed by Narendra and Goldwyn [3] using $V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$ as a Lyapunov function candidate, where P is a constant positive definite matrix ($\mathbf{x}^T P\mathbf{x} > 0, \mathbf{x} \not\equiv 0$, denoted more simply as $P > 0$) and $\mathbf{x}$ is the n-dimensional state vector describing the system behavior. It was shown that a $P > 0$ did exist such that $\dot{V}$ (the time derivative of $V(x)$ evaluated along the system trajectories, $\dot{V} \equiv (\nabla V)^T \dot{\mathbf{x}}$) was negative definite for any $k(\sigma, t) \in (0, \bar{K})$ for all σ and t if the conditions of Theorem 1 were satisfied, thus establishing the stability of the system by Lyapunov's direct method. In this paper, k is assumed to be a function of time only.

The well-known Popov result ([12] or cf [9]) was first generalized to apply to nonlinear time-varying situations by Cho and Narendra [5] using the passive operator technique; a similar result was obtained by the authors [6] using Lyapunov's direct method. In this latter proof, the Lyapunov function candidate was a generalization of the Lure–Postnikov form used by Popov, viz.,

$$V(x, t) = \mathbf{x}^T P\mathbf{x} + 2k(t) \int_0^\sigma f(w)\, dw,$$

where $\sigma \equiv q(D)x$. In the special case that $f(w) = w$, this reduces to $V(x, t) = \mathbf{x}^T P\mathbf{x} + k\sigma^2$. Brockett and Forys [7] originally obtained this theorem in its linear form. It is evident that in the use of this $V(x, t)$, dk/dt enters into V and thus must be constrained by (3.1).

Theorem 3 is a direct generalization of Theorem 2, where the theorem of Lyapunov is supplanted by the special case of a theorem due to Corduneanu (cf. [10], [11]), which replaces the requirement $\dot{V} < 0$ with $\dot{V} \leqq g(t)V$, where $\dot{y} = g(t)y$ must be a stable differential equation. This yields the integral constraint (3.3a), (3.3b) of Theorem 3.

4. Perturbation analysis. In the undamped case, the solutions of the Mathieu equation are unstable for any $q \neq 0$ if $a = 1$. The stability boundary in this neighborhood is to be determined if $\zeta \neq 0$.

The damping ζ ($\zeta \ll 1$) will be used as the perturbation parameter in this analysis. It will be seen that on the boundary, $q_0^* = O(\zeta)$, so define $q \equiv \rho\zeta$. Substitute $x = x_0 + \zeta x_1 + \cdots, a = 1 + \zeta a_1 + \cdots$ into the differential equation and equate terms of equal powers of ζ:

(i) $\zeta^0 : \ddot{x}_0 + x_0 = 0$,

(ii) $\zeta^1 : \ddot{x}_1 + x_1 = [2\rho \cos 2t - a_1]x_0 - 2\dot{x}_0$.

The solution of (i) (the generating solution) is

$$x_0 = B_0 \sin t + C_0 \cos t;$$

substituting into (ii) and using standard trigonometric identities, we have

$$\ddot{x}_1 + x_1 = [-(a_1 + \rho)B_0 + 2C_0]\sin t + [-2B_0 - (a_1 - \rho)C_0]\cos t$$
$$+\rho[C_0 \cos 3t + B_0 \sin 3t].$$

The secular driving terms may be eliminated for some (B_0, C_0) other than the trivial solution $(0, 0)$ only if

$$\det\begin{vmatrix} -(a_1 + \rho) & +2 \\ -2 & -(a_1 - \rho) \end{vmatrix} = a_1^2 - \rho^2 + 4 = 0,$$

thereby ensuring stability to the first approximation. Hence in the neighborhood of $a = 1$, the stability boundary is defined by the parabola

$$(q_0^*)^2 = (a - 1)^2 + (2\zeta)^2. \tag{4.1}$$

It is to be noted that the parabolas are asymptotic to the lines $q = \pm|a - 1|$ which are the stability boundaries for the undamped case to the first approximation. It should be stressed that this boundary is only valid in the small region where $|a - 1| = O(\zeta)$. At $a = 1$, $q_0^* = 2\zeta$.

5. Application of the stability theorems.

Application of Theorem 1. It is necessary to find $\Gamma^* \equiv \min_\omega \Gamma(\omega^2) \equiv \min_\omega \times \{\mathrm{Re}\, G(j\omega)\}$; then by Theorem 1, the solutions of the damped Mathieu equation are stable if $k(t) < -1/\Gamma^*$. By inspection,

$$\Gamma(\omega^2) = \frac{\delta - \omega^2}{(\delta - \omega^2) + (2\zeta\omega)^2}.$$

It can be seen that

$$\frac{d\Gamma}{d\omega} = \frac{2\omega[(\delta - \omega^2)^2 - (2\zeta)^2\delta]}{[(\delta - \omega^2) + (2\zeta\omega)^2]^2},$$

which has zeros at $\omega_1^2 = \sqrt{\delta}(\sqrt{\delta} - 2\zeta)$ and $\omega_2^2 = \sqrt{\delta}(\sqrt{\delta} + 2\zeta)$. By inspecting the sign of $d\Gamma/d\omega$ in the vicinity of ω_1 and ω_2, it can be seen that ω_1 corresponds to a maximum, and ω_2 to a minimum. Thus

$$\Gamma^* = \Gamma(\omega_2^2) = \frac{-1}{4\zeta[\sqrt{\delta} + \zeta]},$$

and it is required that $k(t) < 4\zeta[\sqrt{\delta} + \zeta]$. Define q_1^* to be that value such that

$$\max_t \{k(t)\} \equiv (a - \delta) + 2q_1^* = 4\zeta[\sqrt{\delta} + \zeta];$$

Theorem 1 then guarantees stability for any

$$q < q_1^* \equiv 2\zeta[\sqrt{\delta} + \zeta] + \tfrac{1}{2}(\delta - a).$$

The parameter δ is still unrestricted in the range $(0, a - 2q)$, so it may be chosen to maximize q_1^*. By inspection, $\partial q^*/\partial\delta > 0$ so take δ to be as large as possible. This yields

$$q_1^* = \zeta(a - \zeta^2)^{1/2} \approx \zeta\sqrt{a}, \tag{5.1}$$

which defines the stability boundary in the parameter space (a, q). At $a = 1$, $q_1^* \approx \zeta$, which is just one half of the answer obtained using the perturbation method.

Application of Theorem 2. For this analysis, it is instructive to interpret (3.1) as a phase-plane restriction on $k(t)$; see Fig. 1. It is evident that the phase-plane plot of $(a - \delta) - 2q \cos 2t$ is an ellipse as shown, having as its major axis the vertical line through the point $(a - \delta, 0)$. Thus to obtain the maximum value of q under the constraint (3.1), it is heuristically clear that one should choose

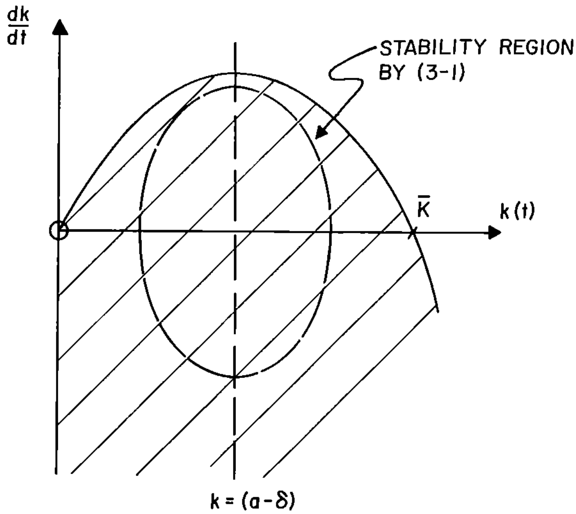

FIG. 1

$\overline{K} = 2(a - \delta)$. This can be shown formally, but it is beyond the scope of this paper to do so. Define λ^* to be that value of λ such that $(G + 1/\overline{K})\cdot(s + \lambda)$ is marginally positive real and q_2^* to be that value of q such that $\dot{k} \leq 2\lambda^* k(1 - k/\overline{K})$; clearly by Theorem 2, the system is stable for all $q < q_2^*$. With $\overline{K} = 2(a - \delta)$, and defining $(a - \delta) \equiv 2\alpha q$, we have

$$\left(G + \frac{1}{\overline{K}}\right) = \frac{1}{4\alpha q}\cdot\frac{s^2 + 2\zeta s + (a + 2\alpha q)}{s^2 + 2\zeta s + (a - 2\alpha q)},$$

where by inspection $1 < \alpha < a/(2q)$. Inspect $h(\omega^2) \equiv$ numerator $\{\mathrm{Re}\,[(G(j\omega) + 1/\overline{K})\cdot(j\omega + \lambda)]$:

$$h(\omega^2) = \omega^4 - 2\omega^2\left[a - 2\zeta^2 - \frac{4\alpha q\zeta}{\lambda^*}\right] + [a^2 - (2\alpha q)^2].$$

It is well known that $x^2 - 2Bx + C \geqq 0$ for all $x \equiv \omega^2 \geqq 0$ if $B^2 = C$, so

$$\left[(a - 2\zeta^2) - \frac{4\alpha q\zeta}{\lambda^*}\right] = (a^2 - (2q)^2)^{1/2}$$

ensures that $G(j\omega)\cdot(j\omega + \lambda^*)$ is marginally positive real. Thus

$$\lambda^* = \frac{4\alpha q\zeta}{(a - 2\zeta^2) - (a^2 - (2\alpha q)^2)^{1/2}},$$

so (3.1) yields the requirement that

$$f(x) \equiv \frac{\sin x}{\alpha^2 - \cos^2 x} \leqq \frac{2q^*\zeta}{(a - 2\zeta^2) - (a^2 - (2\alpha q^*)^2)^{1/2}} \quad \text{for all} \quad x \equiv 2t.$$

It is now necessary to determine $f^* \equiv \max_x f(x)$:

$$\frac{df}{dx} = \frac{[(\alpha^2 - 1) - \sin^2 x]\cos x}{[\alpha^2 - \cos^2 x]^2},$$

which has six zeros: $x_1 = \pi/2$, $x_2 \in (0, \pi/2)$ and $x_3 \in (\pi/2, \pi)$ such that $\sin x_2 = \sin x_3 = (\alpha^2 - 1)^{1/2}$ and three more which need not be considered, as $f(x) < 0$ for those values. It is evident that

(i) x_1 is a maximum only if $\alpha^2 - 1 \geqq 1$ or $\alpha < \sqrt{2}$.

(ii) x_2 and x_3 are equal maxima if $\alpha^2 - 1 < 1$ or $\alpha < \sqrt{2}$.

So

$$f^* = \begin{cases} \dfrac{1}{2(\alpha^2 - 1)^{1/2}}, & 1 < \alpha < \sqrt{2}, \\[2ex] \dfrac{1}{\alpha^2}, & \sqrt{2} \leqq \alpha. \end{cases} \tag{5.2}$$

In order to avoid unnecessary complexity, it is useful to approximate

$$(a^2 - (2\alpha q)^2)^{1/2} \approx a\left[1 - \frac{1}{2}\left(\frac{2\alpha q}{a}\right)^2\right],$$

for which it is required that $(2\alpha q/a)^2 \ll 1$, which will be seen to be valid. This yields

$$\frac{2q_2^*\zeta}{(a/2)(2\alpha q_2^*/a)^2 - 2\zeta^2} = f^*. \tag{5.3}$$

First consider $\alpha \geqq \sqrt{2}$; we desire to maximize q_2^* defined by (5.3) in this range; substituting (5.2) into (5.3) and solving for q_2^* yields

$$q_2^* = \zeta a\left[\frac{1}{2} + \left(\left(\frac{1}{2}\right)^2 + \frac{1}{a\alpha^2}\right)^{1/2}\right].$$

By inspection q_2^* decreases monotonically as α increases, so the maximum must lie in the range $1 < \alpha < \sqrt{2}$. In this range,

$$q_2^* = \frac{\zeta a}{\alpha^2}\left[(\alpha^2 - 1)^{1/2} + \left((\alpha^2 - 1) + \frac{1}{a}\alpha^2\right)^{1/2}\right]$$

and

$$\frac{1}{2}\frac{\partial q_2^*}{\partial \alpha} = \frac{[2 - \alpha^2(1 + 1/a)](\alpha^2 - 1)^{1/2} + [2 - \alpha^2]((1 + 1/a)\alpha^2 - 1)^{1/2}}{\alpha^3((\alpha^2 - 1)((1 - 1/a)\alpha^2 - 1))^{1/2}},$$

which is positive at $\alpha \approx 1+$, negative at $\alpha = \sqrt{2}$ and zero at α_0 defined by

$$\left[\frac{2 - (1 + 1/a)\alpha_0^2}{\alpha^2 - 1}\right]^2 = \frac{(1 + 1/a)\alpha_0^2 - 1}{\alpha_0^2 - 1};$$

the solution $\alpha_0 = (1 + a/(1 + a))^{1/2}$ is in the range $(1, \sqrt{2})$ for all positive a. The corresponding maximum value of q_2^* is

$$q_2^* = \zeta(a(1 + a))^{1/2}. \tag{5.4}$$

Since for large a, $q_2^* \approx \zeta a$, it can be seen that the area of this stability region is much larger than that obtained using the circle criterion; e.g., even at $a = 1$, $q_2^* = \sqrt{2}\,\zeta$. Since $q_2^* = O(a\zeta)$ for $a \geqq O(1)$, and $\alpha < \sqrt{2}$, the assumption that $(2\alpha q/a)^2 \ll 1$ is valid.

Application of Theorem 3. As this theorem is still more complex than the preceding, it will be assumed that $\bar{K} = \infty$ for the sake of simplicity. Hence λ must satisfy

$$\operatorname{Re}\{G(j\omega - \rho)\cdot(j\omega + \lambda - \rho)\} > 0 \quad \text{for all real } \omega.$$

As before, define λ^* to be that value of λ such that $G(s - \rho)\cdot(s + \lambda - \rho)$ is marginally positive real; then q_3^* as that value of q such that

$$\int_0^T g^*(t)\,dt = 0, \quad \text{where} \quad g^*(t) \equiv \sup\left\{-2\rho; \left[-2\lambda^* + \frac{1}{K}\frac{dk}{dt}\right]\right\}.$$

Then for any $q < q_3^*$, Theorem 3 ensures the stability of the solutions of the Mathieu equation.

By inspection, $G(s - \rho)$ is stable if (i) $\rho < \zeta$ and (ii) $\delta > \zeta^2$ and further

$$G(s - \rho)\cdot(s + \lambda^* - \rho) = \frac{s + (\lambda^* - \rho)}{s^2 + (2\zeta - 2\rho)s + (\delta - 2\zeta\rho + \rho^2)}$$

is of the form

$$\frac{s + a}{s^2 + bs + c},$$

where a, b and c are positive if $\lambda > \rho$ and $\rho < \zeta$. This function is marginally positive real if $a = b$, so (iii) $\lambda^* = 2\zeta - \rho$. The requirements (i) and (iii) are satisfied if

$$\rho = (1 - \alpha)\zeta, \qquad \lambda = (1 + \alpha)\zeta, \qquad 0 < \alpha \leqq 1. \tag{5.5}$$

Thus

$$g(t) = \sup_t \left\{ -2(1-\alpha)\zeta; \left[-2(1+\alpha)\zeta + \frac{4q \sin 2t}{a - \delta - 2q \cos 2t} \right] \right\}.$$

Since one may choose $\delta \approx \zeta^2 \ll a$ and it will be seen that $q < q_3^* = O(\zeta a) \ll a$, $g(t)$ may be simplified by setting $a - \delta - 2q \cos 2t \approx a$. Hence

$$g(t) = \sup_t \left\{ -2(1-\alpha)\zeta; \left[-2(1+\alpha)\zeta + \frac{4q}{a} \sin 2t \right] \right\}.$$

The form of $g(t)$ is shown in Fig. 2.

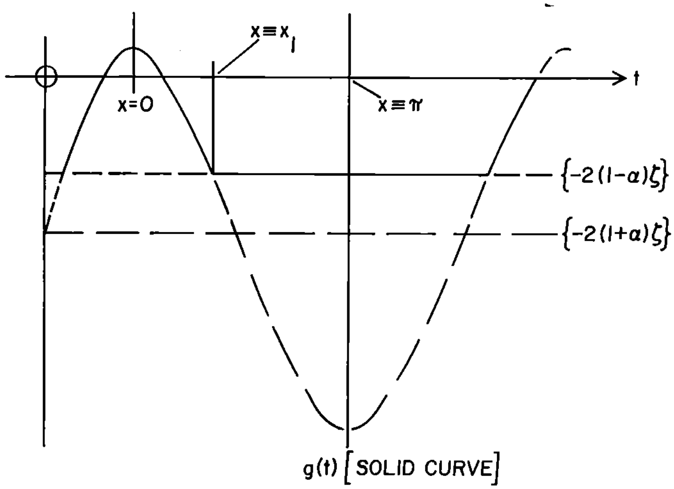

FIG. 2

It is evident that

$$\int_0^T g(t)\,dt \propto I \equiv \int_0^{x_1} \left\{ -2(1+\alpha)\zeta + \frac{4q}{a} \cos x \right\} dx - 2(1-\alpha)\zeta(\pi - x_1), \tag{5.6}$$

where x_1, $0 \leqq x_1 \leqq \pi/2$, is defined by

$$2(1-\alpha)\zeta = 2(1+\alpha)\zeta - \frac{4q}{a} \cos x_1$$

or

$$\cos x_1 = \frac{\alpha \zeta a}{q}.$$

Thus

$$I = -2\zeta\pi + 2\alpha\zeta(\pi - 2x_1) + 4\left(\left(\frac{q}{a}\right)^2 - (\alpha\zeta)^2\right)^{1/2}.$$

Since it is desired to choose α so as to render I as negative as possible, it is necessary to inspect $\partial I/\partial\alpha$:

$$\frac{\partial I}{\partial \alpha} = 2\zeta(\pi - 2x_1) - 4\alpha\zeta\frac{\partial x_1}{\partial\alpha}\frac{4\alpha\zeta^2}{((q/a)^2 - (\alpha\zeta)^2)^{1/2}}.$$

Substituting

$$\frac{\partial x_1}{\partial\alpha} = \frac{-\zeta}{((q/a)^2 - (\alpha\zeta)^2)^{1/2}}$$

yields $\partial I/\partial\alpha = 2\zeta(\pi - 2x_1) \geqq 0$ for all $x_1 \in [0, \pi/2]$, so clearly I is made as negative as possible by setting $\alpha = 0+$. This yields $I \approx 4q/a - 2\pi\zeta$ which is negative for all $q < q_3^*$,

$$q_3^* = \frac{\pi}{2}a\zeta.$$

It would appear that for $a < a_0$ defined by $(a(a+1))^{1/2} = (\pi/a)a$, or $a_0 = 0.681$, Theorem 3 yields a smaller value of q^* than Theorem 2. This occurs because it was assumed that $\overline{K} = \infty$, which is extremely restrictive for $a \approx 1$ or less.

It can be seen, however, that as $\rho \to 0$, the stability conditions of Theorem 3 become those of Theorem 2. Hence since Theorem 2 can be shown to be a special case of Theorem 3, it is justifiable to state that q_3^* cannot be less than q_2^*, or

(5.7)
$$q_3^* = \begin{cases} (a(a+1))^{1/2}, & a \leqq a_0, \\ \dfrac{\pi}{2}a\zeta, & a \geqq a_0. \end{cases}$$

The three stability boundaries are shown in Fig. 3.

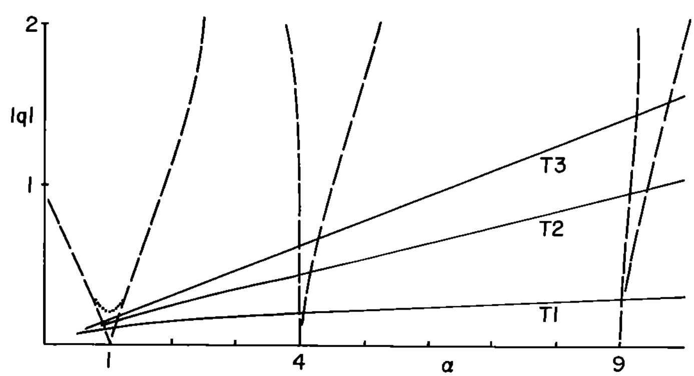

FIG. 3

6. Conclusions. Each theorem presented provides sufficient conditions for the stability of a linear time-varying system of the general form defined in § 2. In applying them to the damped Mathieu equation, which is of quite wide current interest, it is felt that the following contributions have been made:

(i) Although the stability boundaries for $\zeta = 0$ are well established, the critical regions $a = 1, 4, 9 \cdots$ for $\zeta \neq 0$ have been largely unexplored. (McLachlan in [2] has determined a few such curves at $a = 1$ and 4 by investigating the exponential growth of the solutions of $\ddot{x} + [a - 2q \cos 2t]x = 0$, which of course is equivalent to determining the stability of the $\zeta \neq 0$ case).

(ii) Three quite general theorems establishing the stability of linear time-varying systems have been applied, and it has been demonstrated that as the sophistication (and hence computational complexity) increases, the resulting parameter constraints become less strict. This is evident in Fig. 3; thus this study provides a rough measure of "merit" for these theorems.

(iii) It is clear that at $a = 1$, the Corduneanu–Popov theorem gives a result that is $\pi/4$ of the necessary and sufficient stability conditions as found in § 4; this is the best result obtained by using a general theorem of this type to date.

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