INT. J. CONTROL, 1973, VOL. 17, NO. 4, 869–880

On various types of decoupling for time-varying systems†

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[Received 20 April 1972]

This paper deals with the problem of decoupling a class of linear time-varying multivariable systems, based on the defining property that the impulse response matrix of a decoupled system is diagonal. Depending on the properties of the coefficient matrices of the vector differential equation of the open-loop system, the system may be uniformly or totally decoupled. The necessary and sufficient conditions that permit a system to be uniformly or totally decoupled by state variable feedback are given. The main contribution of this paper is the precise definition of these two classes of decoupling and a rigorous derivation of the necessary and sufficient conditions which show the necessity of requiring that the system be of constant ordered rank with respect to observability. A simple example illustrates the importance of having several definitions of decoupling. Finally, the results are specialized to the case of time invariant systems.

1. Introduction

The problem of decoupling time-invariant multivariable systems using state variable feedback was solved partially by Rekasius (1965) and completely by Falb and Wolovich (1967). Further refinements and improvements can be found in the works of Gilbert (1969), Gilbert and Pivnichny (1969), Wonham and Morse (1970) and Silverman and Payne (1970). The results of Falb and Wolovich (1967) were extended to the case of time-varying systems by Porter (1969) and Viswanadham and Venkatesh (1969).

The aims of this paper are to more rigorously define the decoupling problem in terms of the impulse response matrix, and to derive necessary and sufficient conditions that permit a system to be uniformly or totally decoupled by state variable feedback. The definition of each class of decoupling is stated in terms of the impulse response matrix which plays a central role in the subsequent derivations. Yet it should be stressed that the final results (theorems 2 and 3) are dependent only on the coefficients of the state vector differential equation.

2. Basic definitions

The primary system representation considered is specified by the vector differential equation

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t)+B(t)u(t), \\ y(t) &= C(t)x(t), \end{aligned}\right\} \qquad (1)$$

† Communicated by the Authors.

where $x(t)$, an n vector, is the state of the system at time t, $u(t)$, an m vector, is the control input and $y(t)$, an m vector, is the output. $A(t)$, $B(t)$ and $C(t)$ are matrices of order compatible with vectors $x(t)$, $u(t)$ and $y(t)$, continuously differentiable $(2n-1)$, n and $2n$ times respectively.

The impulse response matrix is defined by

$$W(t, \tau) = \left\{ \begin{array}{ll} C(t)\Phi(t, \tau)B(\tau), & t \geqslant \tau, \\ 0, & t < \tau, \end{array} \right\} \tag{2}$$

where $\Phi(t, \tau)$ is the transition matrix associated with $A(t)$. From (2) the response at the output y_i due to an impulse $\delta(\tau)$ at input u_j is defined by

$$W_{ij}(t, \tau) = c_i(t)\Phi(t, \tau)b_j(\tau), \quad t \geqslant \tau, \tag{3}$$

where $c_i(t)$ is the ith row of $C(t)$ and $b_j(\tau)$ is the jth column of $B(\tau)$.

The objective is to decouple (1) by using the feedback law

$$u(t) = F(t)x(t) + G(t)\omega(t), \tag{4}$$

where $\omega(t)$, an m vector, is the reference input. Substitution of (4) into (1) yields

$$\left. \begin{array}{l} \dfrac{dx(t)}{dt} = [A(t) + B(t)F(t)]x(t) + B(t)G(t)\omega(t), \\ y(t) = C(t)x(t). \end{array} \right\} \tag{5}$$

Thus, if the matrices for the closed-loop system are denoted by

$$\left. \begin{array}{l} A^c \triangleq A(t) + B(t)F(t), \\ B^c \triangleq B(t)G(t), \end{array} \right\} \tag{6}$$

the definitions (2) and (3) pertain to the closed-loop system by replacing $A(t)$, $B(\tau)$, $C(t)$, $\Phi(t, \tau)$ and $W(t, \tau)$ by $A^c(t)$, $B^c(\tau)$, $C(t)$, $\Phi^c(t, \tau)$ and $W^c(t, \tau)$ respectively ; in particular,

$$W^c(t, \tau) = \left\{ \begin{array}{ll} C(t)\Phi^c(t, \tau)B(\tau)G(\tau), & t \geqslant \tau, \\ 0, & t < \tau. \end{array} \right\} \tag{7}$$

Various types of decoupling may be defined in terms of the elements of the closed-loop impulse response matrix $W^c(t, \tau)$.

Definition 1

The closed-loop system (5) is uniformly decoupled if

(i) $W_{ij}{}^c(t, \tau) \equiv 0$ for all t and τ, $i \neq j$;
(ii) $W_{ii}{}^c(t, \tau) \neq 0$, for almost all $t \geqslant \tau$ and for all τ.

Definition 2

The closed-loop system (5) is totally decoupled if

(i) $W_{ij}{}^c(t, \tau) \equiv 0$ for all t and τ, $i \neq j$;
(ii) $W_{ii}{}^c(t, \tau)$ is non-zero for almost all $t \geqslant \tau$ and for almost all τ.

These definitions of decoupling are all consistent with the intuitive definition : ' A plant is decoupled if the ith input affects only the ith output '. The various conditions on $W_{ii}^{c}(t, \tau)$ determine the manner in which ω_i affects $y_i(t)$; the system coefficient matrices determine the class of decoupling that may be achieved by state variable feedback. In the case of time-invariant systems all these definitions coincide since the impulse response is a function of $(t-\tau)$.

It may be noted that uniform and total decoupling are not the only two types that may exist in a time-varying system. Other types of decoupling, which are more general than defined above, may exist. The following example demonstrates one such type which does not fall within the ambit of uniform or total decoupling. Consider the decoupled system, $A(t)=\text{diag}\{-a_1, -a_2\}$, $B(t)=\text{diag}\{f_1(t), f_2(t)\}$, $C(t)=I_2$, where f_1, f_2 are any pair of differentiable functions such that $f_1 f_2 \equiv 0$, but there exist mutually exclusive intervals T_1 and T_2 such that $f_1(t) \neq 0$, $t \in T_1$ and $f_2(t) \neq 0$, $t \in T_2$. Note that $W_{ij}(t, \tau) \equiv 0$ for all t and τ, $i \neq j$ and $W_{ii}(t, \tau)=\exp[-a_i(t-\tau)] f_i(\tau)$, $i=1, 2$. Obviously $W_{11}(t, \tau) \equiv 0$ for all $t \geqslant \tau$, $\tau \in T_2$ and $W_{22}(t, \tau) \equiv 0$ for all $t \geqslant \tau$, $\tau \in T_1$. Hence this example will not fall in the domain covered by definitions 1 and 2. In analogy to corresponding types of controllability and observability, systems of this sort can be said to be completely decoupled.

In the following section the conditions under which $W_{ij}^{c}(t, \tau) \equiv 0$ for all t and τ are derived in terms of the coefficient matrices $A^{c}(t)$, $B^{c}(t)$ and $C(t)$. Using these conditions the main theorem is proved.

3. Fundamental lemmas

Define the sequence of matrices $\{S_k\}$,

$$S_0(t) \triangleq C(t)$$
$$\vdots$$
$$S_{k+1}(t) \triangleq S_k(t) A(t) + \frac{dS_k(t)}{dt} \tag{8}$$

and the $(k+1) \times n$ matrices $R_{k,i}$, $i=1, 2, \ldots, m$, $k=0, 1, \ldots, (n-1)$,

$$R_{k,\,i}^{\mathrm{T}} \triangleq [S_{0,\,i}^{\mathrm{T}}, S_{1,\,i}^{\mathrm{T}}, \ldots, S_{k,\,i}^{\mathrm{T}}], \tag{9}$$

where $S_{k,\,i}$ denotes the ith row of S_k. By the previous convention $\{S_k^{c}(t)\}$ and $R_{k,\,i}^{c}(t)$ denote the sequence and corresponding matrix for the closed-loop system. The differentiability assumptions on $A(t)$ and $C(t)$ ensure the existence of $\{S_k^{c}\}$, $k=0, 1, 2, \ldots, n$.

The class of systems treated in this analysis are said to be of constant ordered rank with respect to observability.

Definition 3

A system of the form (1) is of constant ordered rank p_i with respect to observability if p_i is a constant such that

$$p_i = \text{rank } R_{p_i-1,\,i}(t) = \text{rank } R_{p_i,\,i}(t) \text{ for all } t \text{ and } i=1, 2, \ldots, m \,;$$

the matrix $R_{n-1,\,i}$ is said to be of constant ordered rank p_i.

Lemma 1

Let $R^c{}_{n-1,\,i}(t)$ be of constant ordered rank p_i for all t. Then $W_{ij}{}^c(t, \tau)$ is identically zero for all $t \geqslant \tau$ and for all τ if and only if

$$S_{k,\,i}{}^c(\tau) b_j{}^c(\tau) \equiv 0 \quad \text{for all } \tau, \quad k=0, 1, 2, \ldots, (n-1). \tag{10}$$

Proof

Suppose $W_{ij}{}^c(t, \tau) \equiv 0$ for all t and τ. Differentiating the indicated element of (7) repeatedly $(n-1)$ times with respect to t and using (8),

$$\frac{\partial^k W_{ij}{}^c(t, \tau)}{\partial t^k} \triangleq S_{k,\,i}{}^c(t) \Phi^c(t, \tau) b_j{}^c(\tau) \equiv 0 \quad \text{for all } t \text{ and } \tau,$$

$$k=0, 1, 2, \ldots, (n-1). \tag{11}$$

Since (11) holds at $t=\tau$, and $\Phi^c(\tau, \tau) \triangleq I_n$ [the nth-order identity matrix], (10) follows.

To prove the converse, suppose that $S_{k,\,i}{}^c(\tau) b_j{}^c(\tau) \equiv 0$ for all τ, $k=0, 1, 2, \ldots, (n-1)$. If $W_{ij}{}^c(t, \tau)$ is not identically zero for $t \geqslant \tau$ and for all τ, then, as shown in Weiss and Kalman (1965), it is separable into a sum of scalar products of the form

$$W_{ij}{}^c(t, \tau) = \sum_{k=1}^{p_i} \theta_k(t) \psi_k(\tau), \tag{12}$$

where $p_i \leqslant n$. The assumption that the matrix $R^c{}_{n-1,\,i}(t)$ has a constant ordered rank p_i for all t is equivalent to saying that the Wronskian L_θ formed from the solutions $\theta_1, \theta_2, \ldots, \theta_{p_i}$ and their $(p_i - 1)$ derivatives with respect to t is non-singular for all t (Silverman and Meadows 1965).

Taking the partial derivative of (12) p_i times with respect to t yields

$$\frac{\partial^q W_{ij}{}^c(t, \tau)}{\partial t^q} = \sum_{k=1}^{p_i} \frac{\partial^q \theta_k(t)}{\partial t^q} \psi_k(\tau), \quad q=0, 1, \ldots, p_i. \tag{13}$$

Eliminating $\psi_1(\tau), \psi_2(\tau), \ldots, \psi_{p_i}(\tau)$ from (13) yields

$$\frac{\partial^{p_i} W_{ij}(t, \tau)}{\partial t^{p_i}} + \alpha_{p_i}(t) \frac{\partial^{p_i-1} W_{ij}{}^c(t, \tau)}{\partial t^{p_i-1}} + \ldots + \alpha_1(t) W_{ij}{}^c(t, \tau) = 0, \tag{14}$$

where $\alpha_{p_i}(t), \ldots, \alpha_1(t)$ are given by $\alpha_{p_i-\delta} = -\dfrac{\partial^{p_i} \theta_{\delta+1}}{\partial t^{p_i}} L_\theta^{-1}$. $\qquad \delta=0, 1, \ldots, p_i - 1$

and are everywhere differentiable $(n+1-p_i)$ times† with respect to t. Hence the solution of (14) is unique (Kaplan 1962). Only if uniqueness is ensured can it be stated that the necessary and sufficient condition for the solution of (14)

† $L_\theta(t)$ and hence $L_\theta{}^{-1}(t)$ is continuously differentiable $(n+2-p_i)$ times from the differentiability assumptions on $A^c(\tau)$, $B^c(\tau)$ and $C(\tau)$, and the elements of $\left[\dfrac{d^{p_i}\theta_1}{dt} \cdots \dfrac{d^{p_i}\theta_p}{dt^{p_i}}\right]$ are continuously differentiable $(n-p_i+1)$ times. Thus the $\alpha_k(t)$ functions which are expressed as products of these elements must be continuously differentiable $(n+p_i+1)$ times.

to be identically zero for all $t \geqslant \tau$ is that the set of initial conditions [see eqn. (11)] must vanish :

$$\left.\frac{\partial^k W_{ij}^{c}(t, \tau)}{\partial t^k}\right|_{t=\tau} = S_{k,\,i}^{c}(\tau) b_j^{c}(\tau) \equiv 0, \quad k = 0, 1, \ldots, p_i - 1. \tag{15}$$

Although p_i may be less than n, the lemma is proved, since by (14) the higher-order partial derivatives of $W_{ij}^{c}(t, \tau)$ with respect to t at $t = \tau$ must be zero as well. Q.E.D.

For a decoupling procedure to be meaningful, it is also necessary to ensure that the diagonal elements $W_{ii}^{c}(t, \tau)$ must be non-zero except at a finite number of isolated points in t for $t \geqslant \tau$ and for appropriate values of τ, i.e. for almost all τ for total decoupling, or for all τ for uniform decoupling. For a given τ, $W_{ii}^{c}(t, \tau)$ satisfies the stated requirement if the following conditions are met.

Lemma 2

Let $R_{n-1,\,i}^{c}$ be of constant ordered rank p_i for all t. Then for $W_{ii}^{c}(t, \tau)$ to be non-zero for almost all $t \geqslant \tau$ it is necessary and sufficient that at least one of the elements of the sequence $\{S_{k,\,i}^{c}(\tau) b_i^{c}(\tau)\}$, $k = 0, 1, \ldots, n-1$ be non-zero.

Proof

The necessity of this condition follows from lemma 1. In proving sufficiency the notation is simplified by defining $\beta(t) \triangleq W_{ii}^{c}(t, \tau)$ for the value of τ under consideration. By eqn. (14) $\beta(t)$ satisfies a linear homogeneous time-varying differential equation of the form

$$\left[\frac{d^{p_i}}{dt^{p_i}} + \alpha_{p_i}(t) \frac{d^{p_i-1}}{dt^{p_i-1}} + \ldots + \alpha_2(t) \frac{d}{dt} + \alpha_1(t)\ \beta(t)\right] = 0. \tag{16}$$

The condition of lemma 2 provides that at least one member of the set

$$\left[\beta(t), \frac{d\beta}{dt}, \ldots, \frac{d^{p_i-1}}{dt^{p_i-1}} \beta\right] \quad \text{at} \quad t = \tau$$

is non-zero ; it must be shown that this guarantees that $\beta(t)$ is non-zero for almost all $t \geqslant \tau$.

The p_ith-order differential eqn. (16) may be expressed in state vector form with

$$z(t) \triangleq \text{col.} \left[\beta(t), \frac{d\beta(t)}{dt}, \ldots, \frac{d^{p_i-1}\beta(t)}{dt^{p_i-1}}\right];$$

by inspection

$$\dot{z}(t) = M(t) z(t), \tag{17}$$

where $M(t)$ is in the phase variable canonical form,

$$M(t) \triangleq \left[\begin{array}{c|c} 0 & I_{p_i-1} \\ \hline -\alpha_1 & -\alpha_2 \ \ldots \ -\alpha_{p_i} \end{array}\right]. \tag{18}$$

The properties of $\beta(t)$ must be ascertained, so define as a scalar output $\sigma(t) \triangleq h^{\mathrm{T}} z \triangleq z_1$, thus $h \triangleq \text{col.}\ [1, 0, \ldots, 0]$. The pair (h^{T}, M) is uniformly observable (Silverman and Meadows 1965).

Denote the transition matrix of the differential eqn. (17) by $\Phi_M(t, \tau)$; this is a non-singular matrix for all $t \geqslant \tau$. Then for the τ specified in lemma 2,

$$z(t) = \Phi_M(t, \tau) z^\circ, \quad t \geqslant \tau, \tag{19}$$

where z° is the initial condition given by lemma 2 to have at least one non-zero element. Thus, since Φ_M is non-singular, $z^\circ = \Phi_M^{-1}(t, \tau) z(t)$ shows that at every $t \geqslant \tau$ $z(t)$ must have at least one non-zero element. But the uniform observability of (h^{T}, M) implies that it is possible to completely determine the vector $z(t)$ given only $\sigma = z_1(t) \triangleq \beta(t)$, hence $\beta(t)$ cannot be zero on any finite subinterval i.e. $\beta(t) \neq 0$ almost everywhere. Q.E.D.

Note again that lemma 2 is established under the assumption that $R_{n-1,\,i}^{\mathrm{c}}$ is of constant ordered rank, i.e. that L_θ is non-singular. This assumption is directly related to the form of the input–output relation of the decoupled subsystem, as is discussed following the main theorem.

4. Main results

An obvious result based on lemmas 1 and 2 is

Theorem 1

Let $R_{n-1,\,i}^{\mathrm{c}}$ be of constant ordered rank p_i for all t and for all $i = 1, 2, \ldots, m$. Then the system (5) is totally (uniformly) decoupled if and only if

(i) $S_{k,\,i}^{\mathrm{c}}(\tau) b_j^{\mathrm{c}}(\tau) \equiv 0$ for all τ, $i \neq j$, $i, j = 1, 2, \ldots, m$, $k = 0, 1, \ldots, n-1$;

(ii) For each $i = 1, 2, \ldots, m$ at least one of the elements in the sequence $\{S_{k,\,i}^{\mathrm{c}}(\tau) b_i^{\mathrm{c}}(\tau)\}$, $k = 0, 1, \ldots, n-1$, is not zero for almost all τ (for all τ).

Remark

It should be stressed that the main theorem is established under the assumption that $R_{n-1,\,i}^{\mathrm{c}}(t)$ has a constant ordered rank for all i. It will be shown in the next section that a natural assumption on the open-loop system (i.e. assuming that d_i defined in lemma 3 are constants) guarantees this requirement.

4.1. *Uniform decoupling by state variable feedback*

The application of the above theorem is most clear when the system permits uniform decoupling. Necessary and sufficient conditions to be satisfied by $A(t)$, $B(t)$ and $C(t)$ for uniform decoupling to be possible are derived from the foregoing developments. By definition 1, a system is uniformly decoupled if $W^{\mathrm{c}}(t, \tau)$ is diagonal and $W_{ii}^{\mathrm{c}}(t, \tau) \neq 0$ for almost all $t \geqslant \tau$ and for all τ. Hence from eqn. (7) it is clear that $G(\tau)$ must be non-singular for all τ. The following lemma defines the indices d_i and the auxiliary matrix $D(t)$ essential for further development. The indices are in general time dependent, but as explained below it is seen to be necessary to assume that they are constants.

Lemma 3

Define $d_1, d_2, \ldots, d_m$ to be constants such that

$$S_{j,\,i}(\tau) B(\tau) \equiv 0, \quad j < d_i \quad \text{and} \quad S_{d_i,\,i}(\tau) B(\tau) \neq 0 \quad \text{for all } \tau \tag{20}$$

and the matrix $D(\tau)$ by

$$D_i(\tau) \triangleq S_{d_i,\, i}(\tau)B(\tau). \tag{21}$$

Then (i) the indices d_i, the vector $S_{k,\, i}(\tau)$ for $k \leqslant d_i$ and the rank of the matrix $D(\tau)$ are invariant under state feedback, and (ii)

$$\text{rank}\,\{R_{d_i,\, i}(\tau)\} = \text{rank}\,\{R_{d_i,\, i}{}^{c}(\tau)\} = (d_i + 1)$$

for all τ and for $i = 1, 2, \ldots, m$.

Proof

From the definition of the sequence $\{S_k\}$ [eqn. (8)], it is given that $S_{0,\, i}{}^{c}(\tau) = S_{0,\, i}(\tau)$. The proof then may be completed by induction : assume that for some $p \in [1, d_i]$ it is true that $S_{(d_i - p),\, i}{}^{c}(\tau) = S_{d_i - p,\, i}(\tau)$. Then for $(d_i - p + 1)$ by eqns. (6) and (8)

$$S_{d_i - p + 1,\, i}{}^{c}(\tau) = \frac{d}{d\tau} S_{d_i - p,\, i}(\tau) + S_{d_i - p,\, i}[A(\tau) + B(\tau)F(\tau)].$$

Since $(d_i - p) < d_i$, $S_{d_i - p,\, i}(\tau)B \triangleq 0$ by (20) ; hence $S_{d_i - p + 1,\, i}{}^{c}(\tau) = S_{d_i - p,\, i}(\tau)$, which proves that

$$S_{k,\, i}{}^{c}(\tau) = S_{k,\, i}(\tau), \quad k = 0, 1, \ldots, d_i. \tag{22}$$

From $B^c = BG$, it follows that $S_{k,\, i}{}^{c}B^{c} = S_{k,\, i}BG$, $k \leqslant d_i$. Since $G(\tau)$ is non-singular for all τ, the vector $S_{k,\, i}(\tau)B(\tau)$ is a null vector if and only if $S_{k,\, i}{}^{c}(\tau)B^{c}(\tau)$ is zero, hence $d_i = \min\,\{k : S_{k,\, i}{}^{c}(\tau)B^{c}(\tau) \neq 0\}$, which establishes the invariance of this index. Finally, $D_i{}^{c}(\tau) \triangleq S_{d_i,\, i}{}^{c}(\tau)B^{c}(\tau)$, so using (21)

$$D_i{}^{c}(\tau) = D_i(\tau)G(\tau). \tag{23}$$

As $G(\tau)$ is non-singular by assumption, the matrix $D(\tau)$ is non-singular if and only if $D^{c}(\tau)$ is non-singular.

To prove (2), suppose there exists a set of scalar functions $\delta_j(\tau)$, $j = 0, 1, 2, \ldots, d_i$ such that

$$\sum_{j=0}^{d_i} \delta_j(\tau)S_{j,\, i}(\tau) \equiv 0 \quad \text{for all} \quad \tau. \tag{24}$$

The proof is complete if it can be shown that necessarily $\delta_j(\tau) \equiv 0$ for all τ, $j = 0, 1, \ldots, d_i$. Post-multiplying (24) by $B(\tau)$ and using (20) and (21), one gets $\delta_{d_i}(\tau)D_i(\tau) \equiv 0$ for all τ. Since $D_i(\tau) \neq 0$ for all τ, $\delta_{d_i}(\tau) \equiv 0$ for all τ. Eliminating $\delta_{d_i}(\tau)$ from (24) and differentiating with respect to τ, one gets

$$\sum_{j=0}^{d_i - 1} \left\{ \frac{d}{d\tau} (\delta_j(\tau))S_{j,\, i}(\tau) + \delta_j(\tau) \frac{d}{d\tau} (S_{j,\, i}(\tau)) \right\} \equiv 0. \tag{25}$$

Post-multiply (24) by $A(\tau)$, add it to (25) and from (8) one gets

$$\sum_{j=0}^{d_i - 1} \frac{d}{d\tau} (\delta_j(\tau))S_{j,\, i}(\tau) + \sum_{j=0}^{d_i - 1} \delta_j(\tau)S_{j+1,\, i}(\tau) \equiv 0. \tag{26}$$

Post-multiplying (26) with $B(\tau)$ and using (20) and (21), one obtains $\delta_{d_i - 1} \equiv 0$. By repeating this procedure it can be established that $\delta_j(\tau) \equiv 0$, $j = 0, 1, \ldots, d_i$. Q.E.D.

In § 5 it is useful to interpret the matrix $D(\tau)$ in terms of the impulse response matrix $W(t, \tau)$. From lemma 1

$$D(\tau)=\begin{bmatrix} S_{d_1,\,1}(\tau)B(\tau) \\ \\ S_{d_m,\,m}(\tau)B(\tau) \end{bmatrix}=\begin{bmatrix} \dfrac{\partial^{d_1}W_1(t,\,\tau)}{\partial t^{d_1}} \\ \\ \dfrac{\partial^{d_m}W_m(t,\,\tau)}{\partial t^{d_m}} \end{bmatrix}_{t=\tau} \qquad (27)$$

where $W_i(t, \tau)$ is the ith row of $W(t, \tau)$.

Theorem 2 (*uniform decoupling*)

The system (1) may be uniformly decoupled if and only if the indices d_i are constant and $D(\tau)$ (lemma 3) is non-singular for all τ. If

$$\Lambda(\tau) \triangleq \text{diag. } \{\lambda_1(\tau) \ldots \lambda_m(\tau)\}$$

is an arbitrary diagonal matrix that is non-singular for all τ and

$$H(\tau) \triangleq \begin{bmatrix} S_{d_1+1,\,1}(\tau) \\ S_{d_2+1,\,2}(\tau) \\ \vdots \\ S_{d_m+1,\,m}(\tau) \end{bmatrix}, \qquad (28)$$

then $F(\tau)=-D^{-1}(\tau)H(\tau)$ and $G(\tau)=D^{-1}(\tau)\Lambda(\tau)$ are particular matrices which guarantee that the closed-loop system (5) is uniformly decoupled.

Proof

Necessity. If the system (5) is uniformly decoupled by state variable feedback (4), then by definition 1, $W^c(t, \tau)$ is diagonal and $W_{ii}(t, \tau)$ is non-zero for almost all $t \geqslant \tau$ and for all τ. This, by theorem 1, implies that $D^c(\tau)$ is diagonal ; from (23) $D^c(\tau)=D(\tau)G(\tau)=\Lambda(\tau)$. Also, since $W_{ii}{}^c(t, \tau)$ is non-zero for almost all $t \geqslant \tau$ and for all τ, it follows from (7) that $G(\tau)$ must always be non-singular, as previously assumed. Hence $D(\tau)=\Lambda(\tau)G^{-1}(\tau)$ must be non-singular.

Sufficiency. In view of eqns. (6), (8), (21) and (28), plus the definition $F(t) \triangleq -D^{-1}(t)H(t)$, one obtains

$$\begin{aligned} S_{d_i+1,\,i}{}^c(\tau) &= \frac{dS_{d_i,\,i}(\tau)}{d\tau}+S_{d_i,\,i}(\tau)[A(\tau)+B(\tau)F(\tau)] \\ &= S_{d_i,\,1,\,i}(\tau)-D_i(\tau)D^{-1}(\tau)H(\tau) \\ &= 0 \quad \text{for all} \quad \tau. \end{aligned} \qquad (29)$$

Hence by (8) and (29), it follows that

$$S_{d_i+k,\,i}{}^c(\tau)B(\tau)G(\tau)\equiv 0 \quad \text{for all } \tau \quad \text{and for } k\geqslant 1. \qquad (30)$$

From eqns. (29) and (30), plus the condition $S_{k,\,i}{}^c B^c = S_{k,\,i}BG \triangleq 0$, $k=1, 2, \ldots, (d_i-1)$ [eqn. (20) and lemma 3], it follows that each member of the sequence $\{S_k{}^c(\tau)B^c(\tau)\}$ is diagonal, i.e. $S_k{}^c(\tau)B^c(\tau)=\text{diag. }\{\mu_{k,\,i}(\tau),\ i=1, 2, \ldots, m\}$, where $\mu_{k,\,i}(\tau)=0$, $k\neq d_i$ and $\lambda_i(\tau)$, $k=d_i$.

It has been seen earlier that L_θ must be non-singular, i.e. $R_{n-1,\,i}{}^{c}$ should have constant ordered rank. From (29) and part (ii) of lemma 3, it can be easily verified that it has been guaranteed that rank $R_{d_i,\,i}{}^{c} = \text{rank } R_{d_i+1,\,i}{}^{c} = \text{rank } R_{d_i,\,i} = d_i + 1$. Thus by theorem 1 and definition 1 the closed-loop system is uniformly decoupled. Q.E.D.

The specific matrix $F(t)$ defined in theorem 2 is only one of the possible matrices that may be used to achieve a closed-loop system that is uniformly decoupled. The input–output differential equation of the ith sub-system is given by $y_i{}^{d_i+1} = \lambda_i \omega_i$, $i = 1, 2, \ldots, m$. Since d_i is constant, the order of this differential equation is invariant, which stems from the assumption made in lemma 1 that L_θ is non-singular for all t. This is in conformity with the observation that the non-singularity of L_θ assures that the coefficient of the highest derivative of y_i in the input–output differential equation is non-zero for all t (Weiss 1964).

By inspection, a more general input–output relation can be obtained by choosing

$$\left.\begin{aligned} F'(t) &= -D^{-1}(t)H'(t), \\ H_i{}'(t) &\triangleq S_{d_i+1}(t) + \sum_{j=0}^{d_i} m_j{}^{i}(t) S_{j,\,i}(t), \end{aligned}\right\} \tag{31}$$

where $m_j{}^{i}(t)$ are arbitrary coefficients. Then the input–output relation of the ith decoupled sub-system is given by

$$y_i{}^{(d_i+1)} + \sum_{j=0}^{d_i} m_j{}^{i}(t) y^{(j)}(t) = \lambda_i(t)\omega_i(t). \tag{32}$$

4.2. *Total decoupling by state variable feedback*

As in § 4.1 it is again necessary to define the indices d_i and the auxiliary matrix $D(\tau)$. One departure is that certain variables now need only exist almost everywhere which for convenience is denoted by $\tau \notin \{\tau\}_k$, where $\{\tau\}_k$ is some set of countable distinct instants of measure zero.

Lemma 4

Define $d_1, d_2, \ldots, d_m$ to be constants such that $S_{j,\,i}(\tau)B(\tau) \equiv 0$ for all τ, $j < d_i$, and $S_{d_i,\,i}(\tau)B(\tau) \neq 0$ for almost all τ $(\tau \notin \{\tau\}_1)$. Then the indices d_i, the vectors $S_{k,i}(\tau)$, $k \leqslant d_i$ and the rank of $D(\tau)$ when $G(\tau)$ is non-singular are invariant under state variable feedback.

The proof of lemma 4 proceeds as that of part (i) of lemma 3. With indices d_i as defined in lemma 4 it is not guaranteed that $R_{d_i,\,i}$ has a constant rank $(d_i + 1)$ as in the case of lemma 3, so we must assume that $R_{d_i,\,i}$ has a constant rank $(d_i + 1)$ for all t so as to ensure the applicability of lemmas 1 and 2. As before (§ 4.1) this assumption ensures that the closed-loop observability matrices $R_{n-1,\,i}{}^{c}$, $i = 1, 2, \ldots, m$ are of constant ordered rank for all t.

It follows from (7) and the definition of total decoupling that $G(\tau)$ should at least be non-singular for almost all τ. Those instants of time $\{\tau\}_2$ when $G(\tau)$ may be singular need not coincide with $\{\tau\}_1$.

Theorem 3 (*total decoupling*)

The system (1) with $R_{d_i,i}(t)$ of constant rank, $i=1, 2, \ldots, m$ can be totally decoupled if and only if $D(\tau)$ (lemma 4) is non-singular for $\tau \notin \{\tau\}_3$.

Note that these instants $\{\tau\}_3$ where $D^{-1}(\tau)$ does not exist are not necessarily only the instants $\{\tau\}_1$.

Proof

Necessity. If the system (6) is totally decoupled, Theorem 1 implies that $D^c(\tau)=D(\tau)G(\tau)=\Lambda(\tau)=\text{diag. }\{\lambda_i(\tau),\ i=1, 2, \ldots, m\}$, where $\lambda_i(\tau)$ must be non-zero for almost all τ. For $\lambda_i(\tau)\neq 0$ for almost all τ, it is only necessary that $D(\tau)$ be non-singular for almost all τ, since $G(\tau)$ is at least non-singular for almost all τ.

Sufficiency. Let $D(\tau)$ be non-singular except for $\tau\in\{\tau\}_3$. Take

$$\left.\begin{aligned} F(\tau) &= -D^{-1}(\tau)H(\tau), && \tau\notin\{\tau\}_3, \\ G(\tau) &= D^{-1}(\tau)\Lambda(\tau), && \tau\notin\{\tau\}_3, \\ F(\tau) &= G(\tau)=\mathbf{0}, && \tau\in\{\tau\}_3, \end{aligned}\right\} \tag{33}$$

where $H(\tau)$ is as defined in theorem 2. Following the same procedure as in the proof of theorem 2, it can be shown that the assumption that rank $R_{d_i,i}=d_i+1$ guarantees that the system is totally decoupled. Q.E.D.

Note that it is necessary to set $G(\tau)$ equal to the null matrix at the set of instant $\{\tau\}_3$, since it may not be possible to decouple at these times, i.e. for $\tau\in\{\tau\}_3$, the ith input cannot be guaranteed not to affect the jth output, $j\neq 1$ unless externally disconnected by the G matrix. The choice of $F(\tau)=\mathbf{0}$, $\tau\in\{\tau\}_3$ is arbitrary, however.

For the specific $F(\tau)$ and $G(\tau)$ considered above the input–output relation of the decoupled system is given by the differential equations

$$y_i^{(d_i+1)}(\tau)=\lambda_i(\tau)\omega_i(\tau), \quad i=1, 2, \ldots, m, \quad \tau\notin\{\tau\}_3 \tag{34}$$

and the system is unforced at instants $\tau\in\{\tau\}_3$.

Example 1. Consider

$$A(t)=\begin{bmatrix} -[1+\exp(-t)] & -1 & 0 \\ [1+3\exp(-t)] & 0 & -1 \\ -3\exp(-t) & 0 & 0 \end{bmatrix}, \quad B(t)=\begin{bmatrix} 0 & 0 \\ -1 & 1 \\ \exp(-t) & t \end{bmatrix},$$

$$C(t)=\begin{bmatrix} \exp(-t) & 0 & 0 \\ 0 & 0 & \exp(-t) \end{bmatrix},$$

then $d_1=1$, $d_2=0$,

$$D(\tau)=\begin{bmatrix} S_{1,1}(\tau) & B(\tau) \\ S_{0,2}(\tau) & B(\tau) \end{bmatrix}=\begin{bmatrix} \exp(-\tau) & -\exp(-\tau) \\ \exp(-2\tau) & \tau\exp(-\tau) \end{bmatrix}$$

and

$$|D(\tau)| = \exp(-2\tau)[\tau - \exp(-\tau)] = 0 \quad \text{at} \quad \tau = \exp(-\tau) = 0{\cdot}566.$$

Now in this example d_1, d_2 exist according to (20) and are constants, $\tau \in [0, \infty]$, yet $|D(\tau)|$ is zero at $\tau = 0{\cdot}566$, so the system cannot be uniformly decoupled. It can, however, be totally decoupled. The input–output relations of the decoupled subsystems with $F(t)$ and $G(t)$ chosen as in (33) are in the form given in (34) where $\{\tau\}_3 \triangleq \tau = 0{\cdot}566$.

5. Time-invariant systems

For systems having A, B and C constant all the types of decoupling are equivalent. For such systems (7) expressed in Laplace transform notation gives the requirement that $W^c(s) = C[sI - A - BF]^{-1}BG$ must be diagonal and non-singular. From (20) and (21) $d_i \triangleq \min. \{j : c_i A^j B \neq 0,\ j = 0, 1, \ldots, n-1\}$ and $D_i = c_i A^{d_i} B$. The indices d_i always exist if $W_i(s)$ is not a null vector. It was shown (Falb and Wolowich (1967) that the time-invariant version of (1) can be decoupled by state variable feedback if and only if D is non-singular, in accordance with the results of § 4.

When A, B, C are constant matrices, (27) may be interpreted in terms of $W(s)$: defining $S \triangleq \text{diag.} \{s^{d_1+1}, s^{d_2+1}, \ldots, s^{d_m+1}\}$,

$$D = \operatorname*{lt}_{s \to \infty} SW(s) \tag{34}$$

(Gilbert 1969). A simple criterion for decoupling may be obtained from this relation : from (34), it follows that $|D| = \operatorname*{lt}_{s \to \infty} s^q |W(s)|$, where $q \triangleq m + \sum_{i=1}^{m} d_i$, so for $|D| \neq 0$ but finite the denominator order of $|W(s)|$ should be more than its numerator order by exactly q. Note that the d_i's can also be obtained directly from $W_i(s)$: say

$$W_i(s) = \left[\frac{p_{i1}(s)}{q_{i1}(s)}, \ldots, \frac{p_{im}(s)}{q_{im}(s)}\right]$$

and let $n_{ij} = [\text{degree } (q_{ij}) - \text{degree } (p_{ij})]$, $j = 1, 2, \ldots, m$; then necessarily $d_i + 1 = \min_j (n_{ij})$.

Theorem 4 (*decoupling of time-invariant systems*)

The time-invariant version of (1) can be decoupled by state variable feedback if and only if $\beta \triangleq$ degree [denominator of $|W(s)|$] $-$ degree [numerator of $|W(s)|$] satisfies $\beta = \sum_{i=1}^{m} \min_j (n_{ij})$.

A multivariable system can be decoupled if and only if the open-loop transfer function matrix is non-singular (Rekasius 1965) ; the necessary and sufficient condition for decoupling by state variable feedback is more stringent.

6. Conclusions

The principal contribution of this paper is the identification of various types of decoupling for linear time-varying systems, analogous to the concepts of controllability and observability. In essence the existence of the indices d_i and the nature of the determinant of $D(t)$ determine whether the system can be uniformly or totally decoupled using state variable feedback. These results are specialized to the time-invariant case and interpreted directly in terms of the open-loop transfer function matrix.

Acknowledgments

The authors would like to acknowledge the thoughtful review of an earlier version of this paper by Dr. I. H. Mufti of N.R.C., Ottawa, Canada, who pointed out an important technical flaw that has been remedied in lemma 2.

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