Strictly Positive-Real Functions and the
Lefschetz-Kalman-Yakubovich (LKY) Lemma

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A network made up of the lumped passive elements $R$, $L$, and $C$ (resistance, inductance, and capacitance) has a driving point impedance $Z(s)$ that is rational and positive-real, and, conversely, any rational function $Z(s)$ that is positive-real can be realized as the driving point impedance of a passive RLC network. The properties of positive-real functions have thus been exhaustively studied in the evolution of modern network theory. Strictly positive-real functions have not received the same attention, however, and this deficiency has led to a basic lack of clarity in one area of absolute stability theory. A resolution of this difficulty as detailed in [1] is outlined in this letter.

Given $Z(s) = (nsi)/d(s))$ having poles and zeros in the left half plane (Re $s < 0$), the necessary and sufficient condition that $Z(s)$ be positive-real (denoted $Z(s) \in \{PR\}$) is that Re $Z(\omega) > 0$ for all real $\omega$. The corresponding conditions for $Z(s)$ to be strictly positive-real ($Z(s) \in \{SPR\}$) have been given in two forms: $Z(s)$ must have poles and zeros in the open left half plane (Re $s < 0$) and either

$$\Re Z(\omega) > 0, \quad \omega \in (-\infty, \infty)$$

or

$$\Re Z(\omega) > \beta > 0, \quad \omega \in [-\infty, \infty].$$

Equation (1a) is not sufficiently strict [and this has led to a fundamental complication in the Lefschetz-Kalman-Yakubovich (LKY) lemma], while (1b) is too stringent.

First, the proposed definition of a strictly positive-real function is motivated by an appeal to network theory. A strictly positive-real function is said to correspond to the driving point impedance of a dissipative network, i.e., a network composed of resistors, lossy inductors, and lossy capacitors. These latter elements may be represented by $L(s + \epsilon)$ and $C(s + \epsilon)$, corresponding to an ideal inductor in series with $R_L = iL$ and an ideal capacitor in parallel with $G_C = eC$, respectively. Directly, an intuitively reasonable definition of strictly positive-real functions ensues.

**Definition I:** $Z(s) \in \{SPR\}$ if and only if there exists some $\epsilon > 0$ such that $Z(s - \epsilon) \in \{PR\}$.

Thus given any passive RLC network with $Z(s) \in \{PR\}$, a dissipative network is always obtained by substituting $L_i(s + \epsilon)$ and $C_i(s + \epsilon)$ for each $L_i(s)$ and $C_i(s)$ in $Z(s)$, yielding the driving point impedance $Z(s + \epsilon)$, and, conversely, for any $Z(s) \in \{SPR\}$ there must exist some $\epsilon_1 > 0$ such that $0 < \epsilon < \epsilon_1$ guarantees that $Z(s - \epsilon) \in \{SPR\}$ while $Z(s - \epsilon_1)$ is merely positive-real.

This definition leads to an important asymptotic property.

**Corollary to Definition I:** If $Z(s) \in \{SPR\}$ then $\Re Z(i\omega)$ can go to zero no more rapidly than $\omega^{-2}$ as $\omega \to \infty$.

**Proof:** Given $Z(s) = (n(s))/d(s)) \in \{SPR\}$, Re $Z(i\omega) \to 0$ as $\omega \to \infty$ only if order $[n(s)] = \langle \text{order } \{d(s)\} - 1 \rangle$, i.e., if

$$c_n s^{n-1} + \cdots + c_2 s + c_1, \quad c_i > 0, \quad i = 1, 2, \cdots, n$$

$$s^n + a_n s^{n-1} + \cdots + a_2 s + a_1, \quad a_i > 0, \quad i = 1, 2, \cdots, n.$$

By expansion,

$$\Re Z(i\omega - \epsilon) = \Re \frac{n(i\omega - \epsilon)d(-i\omega - \epsilon)}{d(i\omega - \epsilon)d(-i\omega - \epsilon)}$$

$$= \frac{\omega^{2(n-1)}[a_n c_n - c_{n-1} - \epsilon c_n] + \cdots}{\omega^{2(n-1)}}.$$

Thus since Re $Z(i\omega - \epsilon) > 0$ is to be satisfied as $\omega \to \infty$, clearly $(a_n c_n - c_{n-1}) > \epsilon c_n > b_n > 0$ is required, and so Re $Z(i\omega) - b_n/\omega^2$ as $\omega \to \infty$.

**Example (Guillemin [2]):** Several points are clarified by considering the driving point impedance of a network made up of two parallel...
paths, the first a lossy capacitor \( C \) in parallel with \( G \) and the second a lossy inductor \( L \) in series with \( R \). The normalized impedance is

\[
Z_1(s) = \frac{s + c_1}{s^2 + c_2s + c_1}, \quad C = 1, \quad L = \frac{1}{a_1 - c_1(a_2 - c_1)} \quad G = (a_2 - c_1)C \quad R = c_1L.
\]

Directly,

\[
\text{Re} \, Z_1(i\omega) = \frac{(a_2 - c_1)\omega^2 + a_1c_1}{(a_1 - c_1)^2 + (a_2)\omega^2}.
\]

When \( c_1 = a_2 \), note that \( G = 0 \) and \( \text{Re} \, Z_1(i\omega) \to 0 \) as \( \omega \to \infty \) when \( \omega \to -\infty \); if \( c_1 = 0 \) then \( R = 0 \). In both cases \( Z_1(\omega) \) is only positive-real and the network having this driving point impedance cannot be realized with lossy elements. Only in the second case is \( Z_1(i\omega) \) zero for finite \( \omega \) (at \( \omega = 0 \)); if \( c_1 \neq a_2 \) then \( Z_1(i\omega) > 0 \) for \( \omega \in (-\infty, \infty) \), so condition (1a) is not in itself a useful definition of a strictly positive-real function. Also, note that \( Z_1(\omega) \) could not be accepted as a strictly positive-real function using condition (1b) under any circumstances.

Every real rational function that is proper (having no more zeros than poles) may be realized by a quadruple \( \{ \psi, c, A, b \} \) as

\[
Z(s) = \psi + c^T (s - A)^{-1} b
\]

where \( \psi \) is a scalar, \( c \) and \( b \) are \( n \times 1 \) column vectors and \( A \) is an \( n \times n \) matrix. A fundamental network theoretic result (which is central to the solution of the absolute stability problem via the Lyapunov direct method) is the Kalman-Yakubovich lemma [3]. One form of this lemma, due to Lefschetz [4], is especially useful in the stability analysis of nonlinear time-varying systems (cf., Narendra and Taylor [5]).

Lemma 1. Given \( s > 0 \), a matrix \( A \) such that \( |sI - A| \) has only zeros in the open left half plane, a real vector \( c \), a scalar \( \psi \), and an arbitrary real symmetric positive definite matrix \( L \) (\( L = L^T > 0 \)), then a real vector \( b \) and a real matrix \( P = P^T > 0 \) satisfying

\[
A^T P + P A = -q q^T - \delta L
\]

exist if and only if \( \bar{z}(\omega) \) is sufficiently small and \( Z(s) \in \{ \text{SPR} \} \).

Only the constraint \( \text{Re} \, Z(i\omega) > 0 \) was originally required in [4]. In Lefschetz, Meyer, and Wonham [6] it was pointed out that this condition is too lax if \( \psi = 0 \); in that case, the additional requirement \( c^T A b < 0 \) must be imposed. Using phase variable canonical form (as in [4], with no loss in generality), viz.,

\[
\begin{array}{cccc}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
c_1 & c_2 & \cdots & c_n \\
\end{array}
\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
b_1 \\
\end{array}
\begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{array}
\]

direct expansion results in

\[
Z(s) = \psi + c_1 s^{n-1} + \cdots + c_n s + c_1
\]

and \( c^T A b = (c_{n-1} - a_n c_n) \). Thus

\[
\text{Re} \, Z(i\omega) = \psi + \frac{-c^T A b}{\omega^2 n} \omega^{2(n-1)} + \cdots
\]

and the condition \( Z(s) \in \{ \text{SPR} \} \) immediately guarantees that \( c^T A b < 0 \) if \( \psi = 0 \) by the Corollary to Definition 1. Hence Definition 1 obviates the necessity of introducing the seemingly artificial auxiliary condition \( c^T A b < 0 \) explicitly.

Finally, Lemma 1 and Definition 1 make the LKY lemma entirely equivalent to one form of the Kalman-Yakubovich lemma due to Meyer [7]. Given \( Z(s) \in \{ \text{SPR} \} \), define

\[
\hat{Z}(s) = Z(s - e) = \psi + c^T (s - e) I - A)^{-1} b = \psi + c^T (s - \hat{A})^{-1} b
\]

where \( \hat{A} \equiv A + e I \); for \( e > 0 \) sufficiently small, \( \hat{Z}(s) \in \{ \text{PR} \} \), and \( |sI - \hat{A}| \) has zeros only in the open left half plane.

Lemma 2 (Meyer [7, Lemma 1]): Given a matrix \( \hat{A} \) such that \( |sI - \hat{A}| \) has only zeros in the open left half plane, a real vector \( b \) such that \( (A, b) \) is completely controllable, a real vector \( c \), and a scalar \( \psi \); then a real vector \( \hat{q} \), a real symmetric positive semidefinite matrix \( M = M^T > 0 \), and a real matrix \( P = P^T > 0 \) satisfying

\[
\hat{A}^T P + P \hat{A} = -\hat{q} \hat{q}^T - \delta M
\]

exist if and only if \( \hat{Z}(s) \in \{ \text{PR} \} \).

Substituting \( \hat{A} = A + e I \) into (4a) yields

\[
A^T P + P A = -q q^T - \delta (M + 2eP).
\]

Since \( (M + 2eP) \) is symmetric and positive definite, an elementary result of matrix theory is that for any \( L = L^T > 0 \) there exists a \( \delta > 0 \) such that

\[
M + 2eP = \delta L + \hat{M}
\]

and \( \hat{M} = \hat{M}^T > 0 \). From Meyer's proof (it is not entirely obvious here) it is thus always possible to satisfy (3a) with \( q \) satisfying \( q q^T = \hat{q} \hat{q}^T + \hat{M} \).

This new definition of strictly positive-real functions may not be universally useful in every situation where it is necessary to impose a stronger condition than \( Z(s) \in \{ \text{PR} \} \); however, these points demonstrate that it plays an important role in the present context.

REFERENCES


