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GENERAL DESCRIBING FUNCTION METHOD FOR SYSTEMS WITH MANY NONLINEARITIES, WITH APPLICATION TO AIRCRAFT PERFORMANCE

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ABSTRACT

Using sinusoidal-input describing functions (SIDF's) is a well known approach for studying limit cycles in nonlinear systems with one dominant nonlinearity [1,2]. In recent years, a number of extensions of the SIDF method have been developed to permit the analysis of systems containing more than one nonlinearity. In many cases, the nonlinear system models that can be treated by such extensions have been quite restrictive (limited to a few nonlinearities, or to certain specific configurations). Furthermore, some results involved only conservative conditions for limit cycle avoidance, rather than actual limit cycle conditions. The technique described in this paper removes all constraints: Systems described by a general state vector differential equation, with any number of nonlinearities, may be analyzed. In addition, the nonlinearities may be multi-input, and bias effects can be treated.

The general SIDF approach was first fully developed in [3]. Its power and use are illustrated here by application to a highly nonlinear model of a tactical aircraft in a medium-angle-of-attack flight regime [4,5]. Some problems associated with direct simulation (especially "obscuring modes" and the initial condition problem) are also discussed.

INTRODUCTION

The study of limit cycle (LC) conditions in nonlinear systems is a problem of considerable interest in engineering. An approach to LC analysis that has gained widespread acceptance is the frequency domain/sinusoidal-input describing function (SIDF) method [1,2]. This technique, as it was first developed for systems with a single nonlinearity, involved formulating the system in the form

$$\begin{aligned} \dot{\underline{x}} &= F\underline{x} + \underline{g}\mu \\ \sigma &= \underline{h}^T\underline{x} \\ \mu &= \upsilon(t) - \phi(\sigma) \end{aligned} \qquad (1)$$

where $\underline{x}$ is an n-dimensional state vector. The first two relations describe a linear dynamic subsystem with input μ and output σ; the subsystem input is then given to be the external input signal $\upsilon(t)$ minus a nonlinear function of σ. There is thus one single-input/single-output (SISO) nonlinearity, $\phi(\sigma)$, and linear dynamics of arbitrary order that may be represented by the SISO transfer function (in Laplace transform notation) $W(s) = \underline{h}^T(sI-F)^{-1}\,\underline{g}$. This system description is a modern control theoretic reformulation of the more conventional "linear plant in the forward path with a nonlinearity in the feedback path" [1,2].

It is then assumed that the input σ may be essentially sinusoidal, e.g., $\sigma = a \cos \omega t$, and the output approximation

$$\begin{aligned} \phi(\sigma) &\cong \text{Re}\,[\psi_1 \exp(i\omega t)] \\ &\triangleq \text{Re}\,[n_1(a) * a \exp(i\omega t)] \end{aligned} \qquad (2)$$

is made[1]. The fourier coefficient[2] ψ_1 (and thus the "gain" n_1) is generally complex unless $\phi(\sigma)$ is single valued; the real and imaginary parts of ψ_1 represent the in-phase (cosine) and quadrature (-sine) fundamental components of $\phi(a \cos \omega t)$, respectively. The so-called describing function $n_1(a)$ in (2) is "amplitude dependent", thus retaining a basic property of a nonlinear operation. By the principle of harmonic balance, the assumed oscillation -- if it is to exist -- must result in a quasi-linearized system with pure imaginary eigenvalues,

$$|i\omega - F + n_1\, \underline{g}\underline{h}^T| = 0$$

for some value of ω, or by elementary matrix operations

$$W(i\omega) = -1/n_1(a) \qquad (3)$$

Condition (3) is easy to verify using the polar or Nyquist plot of $W(i\omega)$ [1,2]; in addition the LC amplitude a is determined in the process.

It is generally well-understood that SIDF analysis as outlined above is only approximate, so caution is always recommended in its use. The standard caveats that $W(i\omega)$ should be "low pass to attenuate higher harmonics" and that $\phi(\sigma)$ should be "well-behaved" (so that the first harmonic in (2) is dominant) indicate that the analyst has to be cautious.

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1 If $\phi(\sigma)$ is not odd ($\phi(-\sigma) \neq -\phi(\sigma)$) or if $\upsilon(t)$ is not zero, a constant term ("bias" or "D.C. value") must occur in (2); such cases present no difficulty [1,2], but are omitted to simplify the discussion.

2 The usual definition of an SIDF is that $n_1(a)$ is chosen to minimize the mean square error between $\phi(a \cos \omega t)$ and $\text{Re}\,[n_1(a)*a \exp(i\omega t)]$; thus $n_1(a)*a$ is the first fourier coefficient [1,2].

The utility of SIDF analysis for systems with one significant SISO nonlinearity as outlined above has naturally resulted in a number of attempts to generalize the technique to the multiple-nonlinearity case. In most work that preceded [3], only SISO nonlinearities were considered, and bias effects (either due to constant inputs or to "rectification" caused by nonlinear effects) were excluded. Also special model configurations were often assumed. The earlier results are discussed more fully in [5]. The LC analysis approach described in this paper removes all restrictions with respect to model configuration, nonlinearity type, or the presence of biases.

THE GENERAL SIDF LIMIT CYCLE ANALYSIS METHOD

The most general system model considered here is

$$\dot{\underline{x}} = \underline{f}\ (\underline{x},\underline{u}) \tag{4}$$

when $\underline{x}$ is an n-dimensional state vector and $\underline{u}$ is an n-dimensional input vector. Assuming that $\underline{u}$ is a vector of constants, denoted $\underline{u}_o$, it is desired to determine if (4) may exhibit LC behavior.

As before, we assume that the state variables are nearly sinusoidal,

$$\underline{x} \cong \underline{x}_c + \mathrm{Re}\,[(\underline{a}\ \exp\ (i\omega t)] \tag{5}$$

where $\underline{a}$ is a complex amplitude vector and $\underline{x}_c$ is the state vector center value (which is not a singularity, or solution to $\underline{f}\ (\underline{x}_o,\underline{u}_o) = \underline{0}$ unless the nonlinearities satisfy certain stringent symmetry conditions with respect to $\underline{x}_o$). Then we again neglect higher harmonics, to make the approximation

$$\underline{f}(\underline{x},\underline{u}_o) \cong \underline{f}_{DF}(\underline{x}_c,\underline{a},\underline{u}_o) + \mathrm{Re}\ [F_{DF}(\underline{x}_c,\underline{a},\underline{u}_o)\ \underline{a} \cdot \exp(i\omega t)] \tag{6}$$

The real vector $\underline{f}_{DF}$ and the (generally complex) matrix F_{DF} are obtained by taking the fourier expansions of the elements of $\underline{f}(\underline{x}_c + \mathrm{Re}\ \underline{a}\ \exp\ (i\omega t),\ \underline{u}_o)$, and provide the quasi-linear or describing function representation of the nonlinear dynamic relation. The assumed limit cycle exists for $\underline{u} = \underline{u}_o$ if $\underline{x}_c$ and $\underline{a}$ can be found so that

$$\begin{aligned} &\text{(i)} \quad \underline{f}_{DF}(\underline{x}_c,\underline{a},\underline{u}_o) = 0 \\ &\text{(ii)} \quad [i\omega I - F_{DF}(\underline{x}_c,\underline{a},\underline{u}_o)]\underline{a} = 0,\ \underline{a} \neq \underline{0} \end{aligned} \tag{7}$$

(F_{DF} has a pair of pure imaginary eigenvalues, and $\underline{a}$ is the corresponding eigenvector.)

The nonlinear algebraic equations (7) are often difficult to solve. A second-order DE with two nonlinearities (from a two-mode panel flutter model) has been treated easily by direct analysis [6]. An iterative method, based on successive approximation, can be used successfully for more complicated problems such as that described in this paper.

A HIGHLY NONLINEAR AIRCRAFT DYNAMICS MODEL

In a realistic model of the dynamics of a high-performance aircraft at moderate angle of attack, one is confronted with a large number of nonlinearities. These nonlinearities arise from the empirical aerodynamic data for the specific aircraft (aerodynamic coefficients and stability derivatives) and from dynamic and kinematic effects. The state equations for the aircraft motion can be written in body axes as in (8) if small off-diagonal moment-of-inertia terms and nonaxial thrust components are neglected [7]:

$$\dot{\underline{x}} \triangleq \begin{bmatrix} \dot\theta \\ \dot u \\ \dot q \\ \dot w \\ \dot v \\ \dot r \\ \dot p \\ \dot\phi \end{bmatrix} = \begin{bmatrix} q\cos\phi - r\sin\phi \\ (X+T)/m + rv - qw - g\sin\theta \\ ((I_z-I_x)pr + M)/I_y \\ Z/m + qu - pv + g\cos\phi\cos\theta \\ Y/m + pw - ru + g\sin\phi\cos\theta \\ ((I_x-I_y)pq + N)/I_z \\ ((I_y-I_z)qr + L)/I_x \\ p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \end{bmatrix} \triangleq \underline{f}(\underline{x},\underline{u}) \tag{8}$$

The state variables are the aircraft velocity components in body axes (u,v,w), the rotational rates about the body axes (p,q,r), and the pitch and roll Euler angles (θ,ϕ). The parameters g, m, I_x, I_y, I_z denote the acceleration due to gravity and the aircraft mass and moments of inertia, respectively.

The aircraft data and response characteristics are associated with the force and moment components, X, Y, Z, I, M, N; these contributions are expressed in terms of non-dimensional aerodynamic force and moment coefficients, for example,

$$L = \tfrac{1}{2}\rho V^2 S b C_{\ell_T} \tag{9}$$

where ρ represents air density, V is the velocity vector magnitude, and S and b denote reference area and wing span. The aerodynamic coefficients are determined by the aircraft control settings,

$$\underline{u}^T = [\delta_s\ \delta_{sp}\ \delta_{ds}\ \delta_r] \tag{10}$$

which are stabilator, spoiler, differential stabilator and rudder, respectively. In addition, they are highly nonlinear functions of angle of attack and sideslip angle,

$$\alpha = \tan^{-1}\ (w/u) \qquad \beta = \sin^{-1}\ (v/V) \tag{11}$$

In terms of these variables, the force and moment contributions (9) were represented in standard form; for example,

$$\begin{aligned} C_{\ell_T} &= C_\ell(\alpha,\beta) + C_{\ell_\delta}(\alpha)\delta_{ds} + C_{\ell_{\delta_{sp}}}(\alpha)\delta_{sp} \\ &+ C_{\ell_{\delta_r}}(\alpha)\delta_r + \frac{b}{2V}\,[C_{\ell_r}(\alpha)r + C_{\ell_p}(\alpha)p] \end{aligned} \tag{12}$$

The nonlinear terms in (12) were supplied in the form of empirically determined values of the aerodynamic coefficients and stability derivatives at various flight conditions. Based on this information, analytic representations were developed by curve fitting; for example,

$$C_{\ell_p} = -k_{34}(1+k_{35}\alpha+k_{36}\alpha^2) \tag{13}$$

To complete the nonlinear state-vector differential equation (8), the approximations

$$\begin{aligned} V &= \sqrt{u^2+v^2+w^2} \cong u \\ \alpha &= \tan^{-1}(w/u) \cong w/u \\ \beta &= \sin^{-1}(v/V) \cong v/u \end{aligned}$$

are used in most instances. The resulting highly nonlinear model with k_i suitably evaluated in the curve fit relations as in (13) is realistic for the aircraft considered at angles of attack between 15 and 30 deg.

The nonlinearities in (8) which were selected for study are $(-r \sin \phi)$, $(I_z-I_x)pr$, Z, N, and L. These five terms are potentially important in studying lateral-mode oscillations, including possible "wing rock" mechanisms, so they were quasi-linearized; the remaining terms in (8) were handled by small-signal linearization.

THE MULTIVARIABLE LIMIT CYCLE ANALYSIS METHOD

Before LC analysis is undertaken, it is useful to obtain the complete equilibrium or trim condition, i.e., the values of $\underline{x}_0$ and $\underline{u}_0$ that satisfy

$$\underline{f}(\underline{x}_0, \underline{u}_0) = \underline{0} \tag{15}$$

determined according to (8). In a preliminary investigation of aircraft behavior for various flight regimes, small-signal linearization is useful: the (n×n) matrix F_0 defined by

$$F_0 \triangleq \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\underline{x}=\underline{x}_0,\ \underline{u}=\underline{u}_0} \tag{16}$$

determines the dynamic properties of the perturbation equation corresponding to (8). The small-signal eigenvalues, or solutions $\lambda_{0,k}$, k = 1, 2,...,n, to the characteristic equation

$$\det(\lambda_0 I - F_0) = 0 \tag{17}$$

govern the transient response of the aircraft to small perturbations for a fixed control setting $\underline{u}(t) \equiv \underline{u}_0$.

For small α, the eigenvalues given by small-signal linearization are generally moderately well damped, and nonlinear effects may not be important. As α increases damping generally decreases, so the nonlinear effects become critical in determining the behavior of the aircraft, and LC conditions may exist.

The iterative solution of condition (7) precedes as follows: First, assume that an oscillation exists in the system. For the present problem, it is natural to assume that the steady-state angle of attack satisfies

$$\alpha = \alpha_0(1 + \kappa \sin \omega_0 t) \tag{18}$$

where α_0 is moderate and κ is generally less than unity[3]. The assumed frequency, ω_0, is initially the imaginary part of the most lightly damped eigenvalue given by small-signal linearization; ω_0 will be adjusted in the subsequent iterations. The goal of the limit cycle investigation is to determine either that some κ (or several values of κ) exists such that (18) is a valid assumption (limit cycles are predicted), or that no value κ can be found for which (18) is consistent with condition (7) (limit cycles probably are not present). The LC analysis computer program developed for such a determination is doubly iterative, and includes the following additional steps:

Step 0: Start the procedure with $\underline{x}_0$ from (15) and F_0 determined by small-signal linearization (16); set i = 0.

Step 1: Choose a trial value of κ in (18), e.g., κ = 0.1.

Step 2: Based on the assumed oscillation (18) and the current quasi-linear system dynamics matrix F_i, determine the amplitudes of oscillation throughout the system model by finding $\underline{a}_i$ in the steady-state solution

$$\underline{x} = \underline{x}_{c,i} + \text{Re}\ (\underline{a}_i \exp\ (i\omega t)) \tag{19}$$

Step 3: Using the quasi-linear system model, determine the adjusted center $\underline{x}_{c,i+1}$ satisfying

$$\underline{f}_{DF,i}(\underline{x}_{c,i+1}, \underline{a}_i, \underline{u}_0) = \underline{0} \tag{20}$$

which reflects the change in $\underline{x}_c$ caused by the postulated sinusoidal component of $\underline{x}$.

Step 4: Obtain the adjusted quasi-linear system dynamics matrix $F_{DF,i+1}(\underline{x}_{c,i+1}, \underline{a}_i, \underline{u}_0)$ which contains the sinusoidal-component describing function gains for all nonlinearities. Reset i = i+1.

Step 5: Calculate the adjusted frequency, ω_i, which is the imaginary part of the most lightly damped of the new quasi-linear eigenvalues.

Step 6: Check to see if the iterative center determination procedure has converged; if not, return to Step 2; if so, continue to Step 7.

Step 7: Compare the most lightly-damped eigenvalues with those obtained for the previous trial value of κ, denoted κ^- (in the first trial $\kappa^- = 0$, i.e., the eigenvalues are as obtained by small-signal linearization (17)):

- If the pair of eigenvalues near the imaginary axis has crossed the axis, then some value of κ exists in the range (κ^-, κ) such that one pair of the adjusted quasi-linear eigenvalues $\lambda_{i,k}(\kappa)$ is on the imaginary axis -- a limit cycle is predicted. The value of κ, denoted κ_0, can be found by further iteration on κ.
- If the pair of eigenvalues near the imaginary axis remains on the same side of the axis, increment κ (for example, by adding $\Delta\kappa$ = 0.1) and repeat Steps 1 to 7.

Steps 2 to 6 represent an iterative solution of the steady-state conditions for the bias component or "center" of the assumed oscillation; condition (7i) is thereby satisfied. The term center is used to distinguish $\underline{x}_c$ from the equilibrium $\underline{x}_0$ (15). Step 7 is a test to see if condition (7ii) can be met for some κ.

If for a representative set of values of κ (e.g., κ = 0, 0.1, 0.2,..., 1.0) the most lightly damped eigenvalue pair does not cross the imaginary axis, then it is predicted that limit cycles cannot exist for the particular fixed control setting $\underline{u}_0$. Otherwise, the above procedure will iterate to find the value or values

3 Choosing the sinusoidal component amplitude to be $\kappa\alpha_0$ often leads to a convenient normalization. For limit cycle analysis about a zero center value, it would not be appropriate.

of κ which corresponds to probable limit cycle amplitudes.

Some comments and details about the procedure mentioned in Steps 2 and 3 are in order, since they are central to this technique. First, consider the problem of obtaining $\underline{a}_i$ in (19): Given the adjusted equilibrium and quasi-linear system dynamics matrix that are known from the previous iteration, $\underline{x}_{c,i}$ and $F_{DF,i}$ plus an assumed oscillation in one state;

$$x_k = a_k \cos(\omega_i t)$$

(neglecting the bias component for simplicity), it is desired to determine the complex vector of amplitudes, $\underline{a}$, such that $\underline{x} = \mathrm{Re}\,[\underline{a}\exp(i\omega_i t)]$. If ω_i is a natural frequency corresponding to $\dot{\underline{x}} = F_{DF,i}\underline{x}$, then (cf. condition (7))

$$(j\omega_i I - F_{DF,i})\underline{a} = \underline{0}$$

The latter relation serves to define the entire vector $\underline{a}$, given one of its elements, a_k, by deleting one of the equations in (21) and solving the remaining (n-1) equations. The solution $\underline{a}$ for specified a_k is not unique unless ω_i is actually an eigenvalue of $F_{DF,i}$; this will be true only for a value of κ for which limit cycles are predicted. This approach is dealt with in more detail in [4,5].

The nonlinearities given in (8-14) involve many multiple-input terms. The general SIDF format is then

$$f \cong f_{DF}(\underline{x}_i,\underline{a}_i,\underline{u}_o) + \mathrm{Re}\sum_{j=1}^{n} f_j(\underline{x}_i,\underline{a}_i,\underline{u}_o)a_{j,i}\exp(i\omega_i t)$$

where f_{DF} and f_j, j=1,n are the describing function gains. The SIDF term f_{DF} for each nonlinearity thus appears in the quasi-linear system equation for the adjusted center (20), and each gain f_j is used in evaluating the quasi-linear system dynamics matrix F_{DF}. In many instances in this study, the nonlinear relations involve powers of state variables; as an example, the SIDF approximation for $x_1x_2^3$ where x_1 and x_2 are arbitrary state variables is

$$\begin{aligned} f_5(x) = x_1x_2^3 &\cong [x_{c1}^3x_{c2}+\tfrac{3}{2}x_{c2}(x_{c1}r_{22}+x_{c2}r_{12})+\tfrac{3}{8}r_{12}r_{22}] \\ &+[x_{c2}^3+\tfrac{3}{4}x_{c2}r_{22}]\ \mathrm{Re}\ [a_1\exp(i\omega t)] \\ &+[3x_{c1}^2x_{c2}+\tfrac{3}{4}x_{c1}r_{22}+\tfrac{3}{2}x_{c2}r_{12}]\ \mathrm{Re}[a_2\exp i\omega t)] \\ &\triangleq f_{5DF}+f_{5,1}\mathrm{Re}[a_1\exp(i\omega t)]+f_{5,2}\mathrm{Re}[a_2\cdot\exp(i\omega t)] \end{aligned}$$

where, denoting the conjugate of a_j by a_j^*,

$$r_{ij} = \mathrm{Re}\ [a_i a_j^*] \quad i,j = 1,2$$

The above result is obtained by substituting for $\underline{x}$ using (5), applying trigonometric identities and discarding the higher harmonic forms. The quantity f_{5DF} is the (hypotheticcal) fifth element of $\underline{f}_{DF}$ and $f_{5,1}$, $f_{5,2}$ become entries of F_{DF}.

APPLICATION OF THE MULTIVARIABLE LIMIT CYCLE ANALYSIS METHOD

The aerodynamic data curve fits obtained by adjusting the coefficients k_i as in (13) were initially verified by determining the eigenvalues obtained by small-signal linearization, for various trim values of angle of attack. Good agreement with the empirical aero model was obtained; in particular, the Dutch roll mode stability boundary given by small-signal linearization of the curve fit model agreed with that given by the experimental aerodynamic model which showed marginal stability for $\alpha \cong 19.6$ deg. This case ($\alpha = 19.6$ deg) corresponds to the nearly straight-and-level flight condition specified in Table 1; the corresponding control setting $\underline{u}_0$ was therefore chosen for study since small-signal linearization leads to nearly marginal stability and higher-order nonlinear terms thus become critical in determining the aircraft performance. The corresponding eigenvalues associated with the Dutch roll mode are $\lambda_{DR} = 0.0366 \pm 1.52i$, which for small perturbations predicts an unstable response. It should be observed that there is a much slower unstable lateral mode ("lateral phugoid"), with eigenvalues $\lambda_{LP} = 0.0187 \pm 0.131i$. In most instances, a mode which is as slow as the lateral phugoid in the present case is not a concern, so attention is generally restricted hereafter to the behavior of the Dutch roll mode.

Table 1. Selected Equilibrium Condition

STATE VARIABLE (ELEMENT OF $\underline{x}_0$)	VALUE
θ_0	17.46 deg
u_0	81.7 m/sec
q_0	0.296 deg/sec
w_0	29.1 m/sec
v_0	6.04 m/sec
r_0	-0.033 deg/sec
p_0	-0.011 deg/sec
ϕ_0	-5.303 deg

The multivariable LC analysis computer program was then used to find limit cycle conditions. It was found that λ_{DR} is virtually on the imaginary axis, $\lambda_{i,DR} = 4\times10^{-5} \pm 1.495i$ for κ equal to 1.20. Corresponding to this value of κ, the "center" value $\underline{x}_c$ and oscillation component $\underline{a}$ for the state vector are given in Table 2.

Table 2. Center and Predicted Limit Cycle Amplitude for the Stable Limit Cycle

STATE VARIABLE (ELEMENT OF $\underline{x}_c$)	CENTER	Re($\underline{a}$)	Im($\underline{a}$)	UNITS
θ_i	18.35	0.259	-0.234	deg
u_i	80.25	-0.177	0.165	m/sec
q_i	0.174	0.219	0.182	deg/sec
w_i	28.80	-0.810	-0.718	m/sec
v_i	6.14	7.38	0.0	m/sec
r_i	0.792	-1.79	-1.89	deg/sec
p_i	-0.310	-7.35	14.90	deg/sec
ϕ_i	8.55	9.55	5.295	deg

Checking the limit cycle prediction requires that nonlinear simulations of the dynamics specified in (8-14) be performed. Choice of the initial condition for this procedure is critical, because there also exists an unstable mode, a slow spiral mode which for κ =

1.2 is governed by $\lambda_S = 0.0618$. If this mode is excited appreciably, its growth will completely obscure the fast limit cycle that is sought. Fortunately, choosing $\underline{x}(0) \propto \text{Re}\,\underline{a}$ will make the limit cycle in the Dutch roll mode be the dominant mode.

This limit cycle prediction shown in Table 2 was checked by choosing $\underline{x}(0) = 0.8\,\text{Re}\underline{a}$. The resulting time histories of pitch angle θ, y body-axis velocity v, and z body-axis velocity w, are portrayed[4] in Fig. 1. The plot of θ shows that the solutions do diverge very slowly, due to a small unavoidable excitation of the spiral mode. The time histories of v and w show that the dominant Dutch roll mode is very slowly growing for the first 25 sec of the simulation, as would be expected for an initial condition that is slightly interior to the predicted limit cycle. The predicted center value of v is nearly exact, while that for w is in error by about -0.5 m/sec, or about -1.4 percent. Finally, the predicted limit cycle frequency is 1.495 rad/sec, while the observed frequency is 1.497 rad/sec; the agreement is excellent. After 25 sec of simulation, the slow divergence begins to alter the limit cycle that developed in the first part.

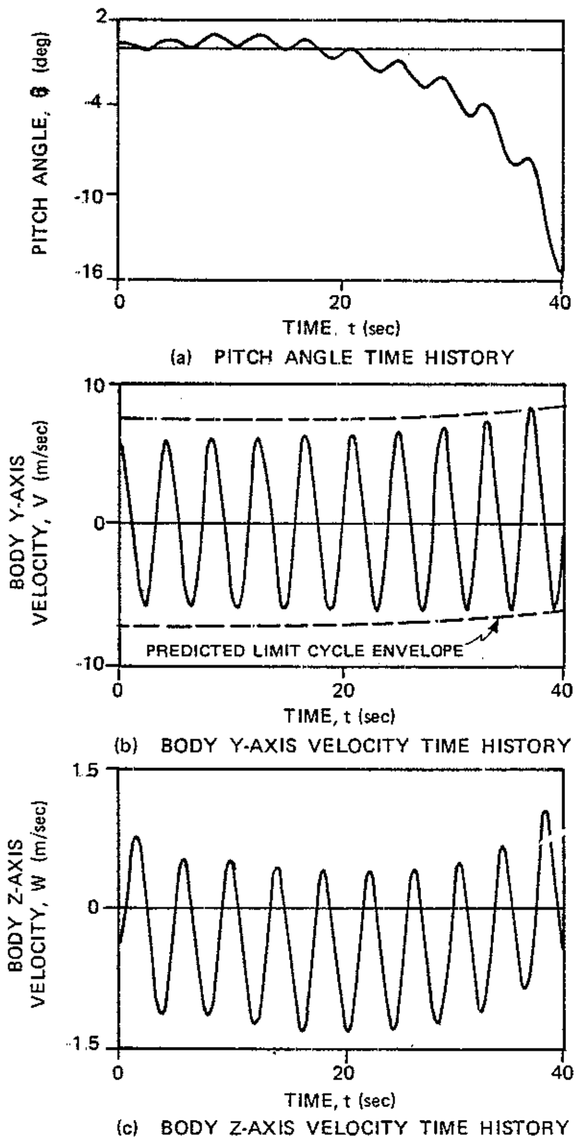

Fig. 1 Simulation of the Limit Cycle Prediction

[4] The plots show the perturbation of each variable about the predicted center value, $x_{c,j}$.

Further analysis of the simulation results was undertaken to attempt to separate out the effect of the slow divergence. The time history depicted in Fig. 1b was processed to determine the exponential growth component ($c_1 e^{c_2 t}$); then the predicted limit cycle envelope is given by the relation

$$e_{LC} = c_1 e^{c_2 t} \pm |a_5|$$

where $|a_5|$ is the amplitude of the predicted limit cycle in v (state 5). This envelope is portrayed in Fig. 1b; within the limits of the simulation accuracy, convergence of the time history to the envelope is shown.

The effort to verify the limit cycle condition by direct simulation has pointed up a major difficulty in using the latter technique as an exploratory tool to locate limit cycles, without recourse to describing function analysis. Realistic aerodynamic models such as those used here often have slow modes that are unstable or that are very lightly damped. Initial conditions for direct simulation must be chosen very carefully to avoid exciting these modes. In a linear system, it is not difficult to use eigenvector information to obtain initial conditions that selectively excite a desired mode. However, eigenvectors are not rigorously defined for nonlinear systems.

The concept which was successfully used in this study may be called the <u>quasi-linear eigenvector</u>; in essence, the complex vector $\underline{a}$, given as in Table 2, is in a sense an amplitude-dependent eigenvector, which specifies an initial condition that excites the predicted oscillation. The fact that the quasi-linear eigenvector $\underline{a}$ is amplitude-dependent is illustrated in Fig. 2, which shows $\underline{a}$ for three values of κ, corresponding to the study depicted in Fig. 1. For κ = 1.0 and 1.5, the eigenvector components for θ and q are too small to be shown; the differences between the remaining components (which are normalized to make the length of the v component equal in each plot) are rather small. For κ = 2.5, the changes in $\underline{a}$ are clearly quite substantial.

SUMMARY AND CONCLUSIONS

The SIDF technique described in this paper permits the investigation of LC conditions in completely general multivariable nonlinear systems. Restrictions as to the type and number of nonlinearities, the system configuration, and the presence of constant inputs have been completely removed.

The study presented here, and the problem solved in [6], illustrate the effectiveness of the general LC analysis method. The predicted LC frequency and "center" value (Fig. 1) are in good agreement with the simulation results; the accuracy of the amplitude prediction is more difficult to assess quantitatively due to the simulation problems mentioned previously (see Fig. 1b). In general, these results bolster the expectation that the iterative LC analysis technique will be found to converge to locate limit cycle conditions, provided that limit cycles indeed exist. Considerable further research could be performed to conclusively demonstrate the power and accuracy of the general SIDF LC analysis approach, and its limitations.

A major point of departure from previous SIDF analysis methods is the substitution of root locus-like plots of "quasi-linear eigenvalues" in lieu of frequen-

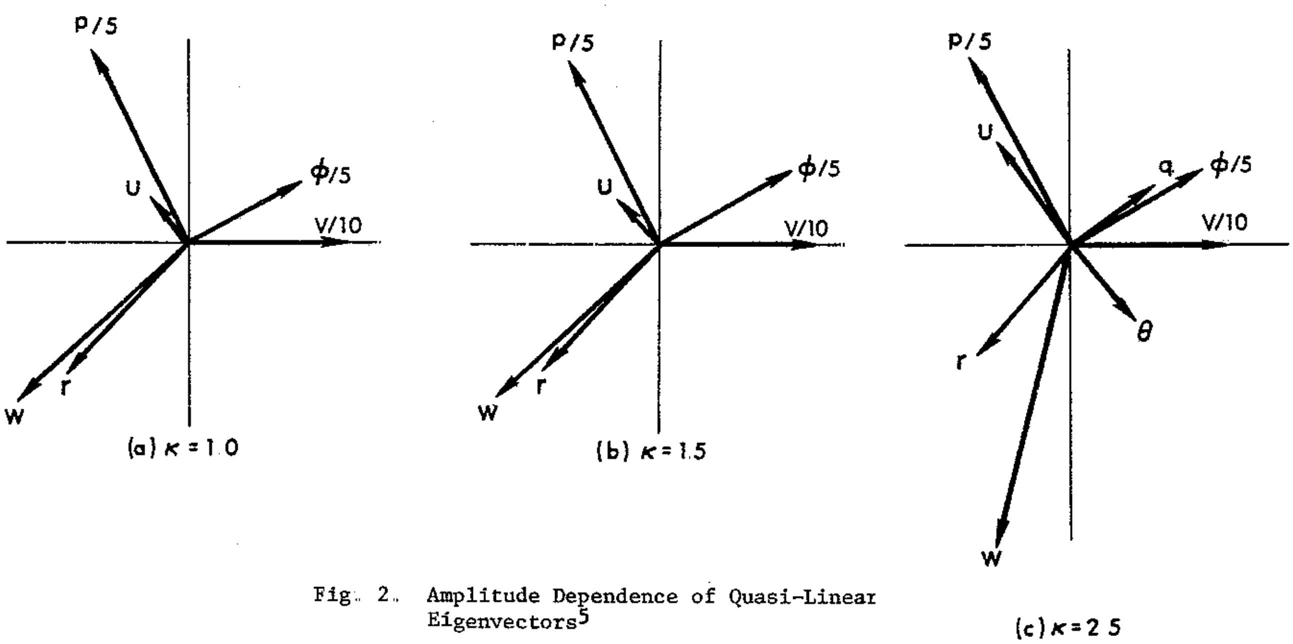

Fig. 2. Amplitude Dependence of Quasi-Linear Eigenvectors[5]

cy domain SIDF techniques for multivariable systems. As a result, one loses the ability to modify or remove LC conditions by the classical methods for altering the frequency response of W(s) by changing pole locations or adding "compensation networks". However, systems designers versed in the more modern technique of pole placement using state variable feedback should find that method of system response compensation applicable to LC conditions found using this new SIDF technique; an approach due to Sankaran [8] appears to be particularly useful in this regard. Combining the general SIDF LC analysis method with an iterative pole position modifying algorithm would result in a very powerful approach to multivariable nonlinear systems synthesis.

Finally, other benefits of this technique are

- Any number of nonlinear effects can be investigated, singly or in any combination, without manipulating the system model into the "linear plant with nonlinear feedback" formulation required in the frequency-domain approach;
- An iterative algorithmic approach to limit cycle analysis is desirable for mechanization on digital computers;
- The amount of computer time required to determine the existence of limit cycles by the general SIDF approach should generally be significantly less than the computer time expenditure that would be needed using direct simulation alone.

The last observation is based on the difficulty of choosing the direct simulation initial condition correctly to excite only the desired nearly oscillatory mode, as discussed in the preceding section.

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5 The eigenvectors correspond to the variables θ, u, q, w, v/10, r, p/5, ϕ5; this scaling was performed to permit all components of $\underline{a}$ to be shown on the plots for κ = 2.5.

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