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Efficient Evaluation of Error Probabilities for Systems with Interference and Gaussian Noise *

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Abstract

A novel, computationally efficient and very accurate method for the calculation of error probabilities in systems with interference and Gaussian noise is presented. The main idea is to approximate the natural logarithm of the Q-function by a truncated version of its Taylor series. As a result, $Q(x)$ can be expressed as a finite product of exponential functions. This enables us to find true and approximate upper bounds for the probability of error by evaluating exponential moments of the interference provided that the individual interference components are mutually independent. The described method is very accurate. In numerous examples, the relative errors between the true probability of error and the approximations did not exceed 1% for systems with an "open eye" and 100% when the eye was closed. Additionally, the method is very effective and easy to use, outperforming most published methods in both the number of computations required and simplicity.

1. Introduction

It is important in the analysis and design of digital communication systems to determine the system performance. The most intuitive and important performance criterion is the probability of error. A nonzero error probability is, due to system imperfections, caused by a noise component in the receiver output signal. The probability distribution of the noise signal is not Gaussian for most receivers, including the MMSE equalizer. Even for zero-forcing equalizers with an ideally normal distributed noise component, system imperfections such as erroneous channel estimation, finite-length filters or a non-ideal sampling time may cause an interference component in the output signal, which is not Gaussian. In addition, a Gaussian approximation for the interference has been shown in many cases to lead to significantly inaccurate results for the error probability. Thus, it is desirable to describe the statistical properties of the interference accurately.

The most straightforward approach to calculate the error probability is the *truncated pulse train approximation* [1]. While small interference pulses are neglected, all possible combinations of the dominant interference samples are evaluated in order to calculate the probability density of the interference. This method is, however, not efficient as the required amount of operations grows exponentially with the number of interference samples considered.

Bounding the error probability is more efficient. The worst case bound [1] always assumes the largest possible amount of interference. Although leading to a very simple expression, this bound is in most cases rather loose. The Chernoff bounds by Saltzberg [2] and Lugannani [3] are easy to compute and yield much better results, but they are still very often loose by more than one order of magnitude. Other bounds like the worst possible distribution bounds [4] and the moment space bounds [5] give better but still not very accurate results. Very tight are the multidimensional moment bounds [6]. Their computation is, however, extremely complicated [7].

Approximating the error probability is another approach. Series expansions [8, 9] yield very good results, but in some cases, the terms in the expansion tend to oscillate. By numerical calculation of an inverse Laplace integral [7], the error probability can be calculated as accurately as desired. Other work which is based on a Fourier series expansion [10] yields almost the same accuracies with a comparable computational effort. A very efficient method leading to an accurate approximation has been developed by Yue [11].

A class of error probability bounds based on an approxi-

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mation of the Gaussian probability distribution is presented in this paper. Given is a real signal consisting of a signal component, interference and noise. The objective is to find the probability that the signal exceeds the optimal decision threshold. Three crucial assumptions are made: Firstly, the noise is a random variable with Gaussian (normal) probability distribution and zero mean; secondly, the noise is uncorrelated with both signal component and interference; and finally, the data symbols of the signal and interference are mutually independent and zero mean.

Section 2 introduces the problem and the basic approach. In Section 3, the Taylor series of $\ln Q(x)$ is truncated after the linear term. This results in a true upper bound on the error probability and corresponds to a method briefly outlined by McGee [12]. The general first-order approximation requires full knowledge of the interference weights h_i and leads to a tight upper bound. In certain situations, only the energy of the total interference may be known, while the values of the h_i are not available. For this case, a simpler albeit looser upper bound is described in Section 3.1.

A very good approximation to the error probability is obtained by taking the quadratic term of the Taylor series expansion into account (Section 4). This second-order approximation is almost exclusively larger than the true error probability and will be referred to as "approximate upper bound". Numerical results for our approximations are shown in Section 5 and Section 6 provides a brief conclusion.

Define for later use: $\mathbb{Z} \triangleq \{0, \pm 1, \pm 2, \ldots\}$ the set of all integer numbers; $\mathbb{N} \triangleq \{1, 2, 3, \ldots\}$ the set of all positive integers; $\mathcal{I}_M \triangleq \{1, 2, 3, \ldots, M\}$ the set of integer numbers between 1 and M; $\mathbb{R}$ the set of real numbers.

2. Problem Formulation and Approach

Any sample of the baseband signal provided to the decision element of a communications receiver can typically be expressed in the form

$$\tilde{\alpha}_0 = \sum_{i=-\infty}^{\infty} \alpha_i h_i + \zeta_0 \tag{1}$$

where $\alpha_0 h_0$ is the signal component, the terms $\alpha_i h_i \ \forall i \neq 0$ $(i \in \mathbb{Z})$ are interference and ζ_0 is zero mean Gaussian distributed noise with variance $\mathcal{E}_\zeta$. The transmitted data symbols α_i are assumed to be produced by pulse amplitude modulation (PAM) with an *even* number of L_i levels. Without loss of generality, we choose the random variables α_i from the finite set of odd integer numbers

$$\mathcal{A}_i \triangleq \{\pm 1, \pm 3, \ldots, \pm(L_i - 1)\}. \tag{2}$$

Furthermore, it is assumed that the α_i take on these values with equal probability, that the data symbols are mutually independent and that they are uncorrelated with the Gaussian noise.

This model applies to a single user system which transmits data symbols α_i at a period of T seconds through a system with overall impulse response $h(t)$. Normally, the number of modulation levels is the same for all symbols, i.e. for this case we get $L_i = L, \ \forall i \in \mathbb{Z}$. At the detector, the received signal is sampled at the time instants $t = t_0 + nT$ ($n \in \mathbb{Z}$). Under the assumption of a stationary channel, the error probability may be calculated from any signal sample. For simplicity, we consider the time $n = 0$ and define $h_i \triangleq h(t_0 - iT)$, which results in the above model (1).

In addition, Equation (1) describes more complicated systems. For example, the output signal of a multiple-input multiple-output linear equalizer employed in a multiuser system may also be expressed in form (1). For that, it is assumed that the symbol rates of all users are identical and that the modulation scheme of the m-th user is quadrature amplitude modulation (QAM) consisting of two independent, in phase quadrature modulated PAM signals with $L_{m,1}$ and $L_{m,2}$ levels, respectively.

Let us, for reasons explained later, normalize the interference samples h_i ($i \in \mathbb{Z} \setminus 0$) by the standard deviation $\sqrt{\mathcal{E}_\zeta}$ of the Gaussian noise component ζ_0 and map them bijectively into a new sequence f_k ($k \in \mathbb{N}$) such that the magnitudes of f_k are nonincreasing. The corresponding data symbols α_i are mapped into the sequence d_k, so that

$$\{h_i | i \in \mathbb{Z} \setminus 0\} \mapsto \left\{ f_k = \frac{h_i}{\sqrt{\mathcal{E}_\zeta}} \middle| f_k^2 \geq f_{k+1}^2, \forall k \in \mathbb{N} \right\} \tag{3}$$

$$\{\alpha_i | i \in \mathbb{Z} \setminus 0\} \mapsto \{d_k = \alpha_i | k \in \mathbb{N}\} \tag{4}$$

$$f_0 = \frac{h_0}{\sqrt{\mathcal{E}_\zeta}} \tag{5}$$

$$d_0 = \alpha_0. \tag{6}$$

The normalized decision variable is then

$$\frac{\tilde{\alpha}_0}{\sqrt{\mathcal{E}_\zeta}} = d_0 f_0 + z + \bar{\zeta}_0 \tag{7}$$

where $\bar{\zeta}_0 \triangleq \zeta_0 / \sqrt{\mathcal{E}_\zeta}$ is the normalized Gaussian noise random variable with zero mean and unit variance and the *interference random variable* is defined as

$$z \triangleq \sum_{k=1}^{\infty} d_k f_k = \frac{1}{\sqrt{\mathcal{E}_\zeta}} \sum_{\substack{i=-\infty \\ i \neq 0}}^{\infty} \alpha_i h_i. \tag{8}$$

The individual data symbols d_k are zero mean, mutually independent random variables with variance

$$\mathcal{E}_{d,k} = \frac{1}{L_k} \sum_{i=1}^{L_k} (2i - L_k - 1)^2 = \frac{1}{3} (L_k^2 - 1). \tag{9}$$

Note that the data symbols originate in a multiuser system from several transmitters, which may employ QAM/PAM schemes with a different number of levels. Thus, the L_k may not be the same for different k.

If $\tilde{\alpha}_0/\sqrt{\mathcal{E}_\zeta}$ is the input to the decision device, the optimal slicing levels are even multiples of the zeroth channel sample [13]: $\{0, \pm 2f_0, \pm 4f_0, \pm(L_k - 2)f_0\}$. The probability of exceeding the decision threshold in the positive direction is then given by

$$\begin{aligned} P_{\text{ex}} &= \text{Prob}\left\{\frac{\tilde{\alpha}_0}{\sqrt{\mathcal{E}_\zeta}} > (d_0+1)f_0\right\} \\ &= \text{Prob}\left\{\bar{\zeta}_0 > f_0 - z\right\} \\ &= \int_{-\infty}^{\infty} Q(f_0 - z)\, p_z(z)\, dz \end{aligned} \tag{10}$$

where $Q(x) \triangleq 1/\sqrt{2\pi}\int_x^\infty \exp(-u^2/2)\,du$ is the complementary cumulative distribution function of the Gaussian noise $\bar{\zeta}_0$ and $p_z(x)$ is the probability density function of the interference z. Due to the symmetry of both Gaussian noise and interference, the probability of error is very well approximated by $2P_{\text{ex}}$. Denoting the expectation operator by 'E', Equation (10) can equivalently be expressed in the form [14]

$$P_{\text{ex}} = E\left[Q\left(f_0 - z\right)\right]. \tag{11}$$

The problem is that Equation (11) cannot be solved in this form. In general, it is not possible or extremely difficult to determine the expectation taken over a nonlinear function of the interference random variable. There are, however, some special nonlinear functions for which this is feasible or for which a closed form expression exists. For example, it is possible to determine the the moments $E[z^n]$ ($n \in \mathbb{N}$) with moderate computational effort. An even simpler solution can be obtained for the exponential moment $E[e^z]$. The problem might therefore be solved by replacing the Q-function in Equation (11) with another nonlinear function for which the expectation can be determined. The method described here approximates the Q-function by a product of exponentials in z.

It follows from Equation (10) that $Q(f_0 - z)$ has to be approximated accurately only in the interval in which $p_z(z)$ is supported, i.e. only for $-D \le z \le D$, where $D \triangleq \sum_{k=1}^\infty (L_k - 1)|f_k|$ denotes the peak distortion of the interference. One may, for example, approximate $Q(f_0 - z)$ reasonably well within the whole interval $|z| \le D$ (Chebyshev polynomial, Fourier series). Another possibility is to use a locally optimal approximation around a point z_0 such that the approximation error vanishes for $z = z_0$ and grows with increasing distance from z (Taylor series). Note in this context that $p_z(z)$ is even symmetrical around $z = 0$ and decreases, on average, strongly with increasing distance from the origin. On the other hand, $Q(f_0 - z)$ increases strongly between $z = 0$ and $z \lesssim f_0$. Assume that the eye is open ($D < f_0$) and that the product $Q(f_0 - z)\, p_z(z)$ is maximal at or close to $z = z_0$. It can then be shown that, with growing distance from $z = z_0$ into either direction, the decreasing function will dominate over the increasing one such that the product $Q(f_0 - z)\, p_z(z)$ vanishes eventually. This behavior suggests to perform a locally optimal approximation of $Q(f_0 - z)$ around z_0 since these values contribute by far the most towards the integral (10). Conversely, less accuracy is necessary with growing distance from z_0, where the product $Q(f_0 - z)\, p_z(z)$ becomes increasingly negligible. Thus, we will consider a locally optimal approximation of $Q(f_0 - z)$ using a Taylor series approach.

Exponential Product Form of $Q(x)$. The natural logarithm of the Q-function may be expanded into a Taylor series: $\ln Q(x) = \sum_{n=0}^\infty c_n\,(x - x_0)^n$ where $x_0 \in \mathbb{R}$ is arbitrary. Taking the exponent of the Taylor series yields

$$Q(x) = Q(x_0) \prod_{n=1}^{\infty} e^{c_n (x - x_0)^n}. \tag{12}$$

The coefficients c_n are given by

$$c_n = \frac{1}{n!}\,\frac{d^n}{dx^n} \ln Q(x)\bigg|_{x = x_0} \tag{13}$$

where '$n!$' is the factorial of n. The product form of $Q(x)$ serves as the starting point for the first- and second-order approximations treated in Sections 3 and 4.

Exponential Moment of the Interference. The *exponential moment* $E[e^{\Lambda z}]$ of the interference random variable needs to be determined for later use. The parameter Λ is in general a complex number, which can be expressed in terms of its real and imaginary components: $\Lambda = \lambda + j\mu$. Substituting Definition (8) into the exponential moment and using the fact that the symbols d_k are mutually independent random variables for all $k \in \mathbb{N}$, we get [15]

$$E\left[e^{\Lambda z}\right] = \prod_{k=1}^{\infty} E\left[e^{\Lambda d_k f_k}\right]. \tag{14}$$

The symbols d_k assume each element in the set (2) with equal probability. After some calculations, the individual exponential moments can be expressed by a ratio of hyperbolic sines:

$$E\left[e^{\Lambda d_k f_k}\right] = \frac{1}{L_k}\,\frac{\sinh\left(L_k \Lambda f_k\right)}{\sinh\left(\Lambda f_k\right)}. \tag{15}$$

Alternatively, the individual exponential moments can be bounded from above. Saltzberg [2] has found an upper bound on the sum of hyperbolic cosines:

$$\frac{2}{L_k}\sum_{i=1}^{L_k/2}\cosh[(2i-1)\lambda f_k] < \exp\left(\frac{1}{2}\,\mathcal{E}_{d,k}\lambda^2 f_k^2\right) \quad (16)$$

where $\lambda \in \mathbb{R}$ and $\mathcal{E}_{d,k}$ is the variance of d_k (Equation (9)). Using Saltzberg's approximation, an upper bound for the individual exponential moment is given by

$$E\left[e^{\Lambda d_k f_k}\right] < \exp\left(\frac{1}{2}\,\mathcal{E}_{d,k}\lambda^2 f_k^2\right) \quad (17)$$

where λ is the real part of Λ.

3. First-Order Approximation

For the remainder of this section, only the term $n = 1$ in the product of Equation (12) is kept while all other terms are neglected. It can be shown, [12], that this first-order approximation is a strict upper bound for the Q-function:

$$Q(x) \leq Q(x_0)\,e^{c_1(x-x_0)}. \quad (18)$$

Setting $x = f_0 - z$, $x_0 = f_0 - z_0$, $c_1 = -\lambda$ and substituting the above expression into Equation (11) yields the probability

$$P_{\text{ex}} < Q(f_0 - z_0)\,e^{-\lambda z_0}\,E\left[e^{\lambda z}\right] \quad (19)$$

where $\lambda = -c_1$ is obtained from Equation (13):

$$\lambda = -\frac{Q'(f_0 - z_0)}{Q(f_0 - z_0)}. \quad (20)$$

$Q'(x)$ is the first derivative of $Q(x)$ and z_0 is a parameter that can be chosen arbitrarily. It will be determined later such that the tightest bound is obtained.

Equation (19) contains the exponential moment of the interference. As shown in Equation (14), it can be expressed as a product of the individual exponential moments. An exact expression for the individual moments has been derived in Equation (15). Generally, this relationship may be used for all interference components. In some cases, however, the number of interference components is large, and the computational effort required when considering all of them may be rather high. In order to reduce the computational load, the contribution from small interference samples may be upper bounded. A good choice is Saltzberg's approximation (17), which is rather loose if the exponent $\lambda d_k f_k$ is large. However, the smaller the magnitude of the exponent, the tighter the bound. For the task of bounding small components f_k, it turns out to be an excellent approximation. Therefore, the set of interference samples will be divided into two groups: One with relatively large magnitudes and the other with small ones.

The reason for reorganizing the interference sequence in nonincreasing order (Transformation (3)) becomes now obvious. Consider that there are M large interference terms f_k ($k \in \mathcal{I}_M$). For these terms, the exact expression for the individual exponential moment (15) will be used. The remaining interference contributions are assumed to be sufficiently small such that their individual exponential moments are very well approximated by the bound in Equation (17). Following Equation (14), the exponential moment of the interference may be upper bounded by

$$E\left[e^{\lambda z}\right] < \exp\left(\frac{1}{2}\mathcal{E}_{z,M}\lambda^2\right)\prod_{k=1}^{M}\frac{\sinh\left(L_k\lambda f_k\right)}{L_k\,\sinh\left(\lambda f_k\right)} \quad (21)$$

where $\mathcal{E}_{z,M}$ is the combined energy of the small interference components:

$$\mathcal{E}_{z,M} \triangleq \sum_{k=M+1}^{\infty}\mathcal{E}_{d,k}f_k^2. \quad (22)$$

Substituting Equation (21) into (19) results in the *general first-order upper bound* $F_1(z_0) > P_{\text{ex}}$, where

$$F_1(z_0) \triangleq Q(f_0 - z_0)\,\exp\left(\frac{1}{2}\,\mathcal{E}_{z,M}\lambda^2 - \lambda z_0\right)\times \prod_{k=1}^{M}\frac{\sinh\left(L_k\lambda f_k\right)}{L_k\,\sinh\left(\lambda f_k\right)}. \quad (23)$$

Note that this bound is valid for all values $z_0 \in \mathbb{R}$.

Tightest Upper Bound With some effort, it can be shown that the first derivative of $F_1(z_0)$ is

$$\frac{dF_1(z_0)}{dz_0} = F_1(z_0)\,\frac{d\lambda}{dz_0}\,g(z_0) \quad (24)$$

where $g(z_0)$ is defined as

$$g(z_0) \triangleq \mathcal{E}_{z,M}\lambda - z_0 + \sum_{k=1}^{M} f_k\,\left[L_k\,\coth(L_k\lambda f_k) - \coth(\lambda f_k)\right]. \quad (25)$$

Lemma 3.1 (a) $F_1(z_0) > 0, \quad \forall z_0 \in \mathbb{R}$;
(b) $\lambda > 0, \quad \forall z_0 \in \mathbb{R}$;
(c) $d\lambda/dz_0 < 0, \quad \forall z_0 \in \mathbb{R}$;
(d) $g(z_0 + \delta z_0) < g(z_0), \quad \forall \delta z_0 > 0,\ z_0 \in \mathbb{R}$;
(e) $g(z_0) = 0$, *for one and only one* $z_0 \in \mathbb{R}$.

A proof for this lemma is provided in the dissertation of Schlagenhaufer [16].

By applying the results of Lemma 3.1 to Equation (24), it can easily be shown that the global minimum of the function $F_1(z_0)$ is located at the point $z_0 = \bar{z}_0$. $\bar{z}_0$ is the solution of the transcendental equation $g(z_0) = 0$. The final result is stated in the following theorem.

Theorem 3.1 *The tightest first-order upper bound for the threshold probability is*

$$P_{ex} < F_1(\bar{z}_0) \tag{26}$$

where the optimal parameter $\bar{z}_0$ *satisfies the condition*

$$g(\bar{z}_0) = 0. \tag{26 b}$$

After numerically solving the transcendental Equation (25) for the root $\bar{z}_0$, the optimal parameter λ is determined with Equation (20) and both values are substituted into the upper bound (23) for the best approximation. In order to guarantee convergence for the iterative solution of $g(\bar{z}_0) = 0$, the bisection, "Pegasus" or a similar method should be applied which encloses the desired root between two values [17].

3.1. Energy Upper Bound

The general first-order bound of Theorem 3.1 becomes tighter the more individual exponential moments are calculated by the exact expression in Equation (15), i.e. the larger the value of M is chosen. This requires, on the other hand, explicit knowledge of the interference weights f_k for all $k \in \mathcal{I}_M$, which may not be available in some situations. If only the respective energies of interference and Gaussian noise are known, the special case $M = 0$ could be considered. Under this condition, the upper bound of Theorem 3.1 reduces to

$$P_{\text{ex}} < Q(f_0 - \bar{z}_0)\, \exp\left(-\frac{\bar{z}_0^2}{2\mathcal{E}_{z,0}}\right) \tag{27}$$

where the optimal parameter $\bar{z}_0$ is the solution of the transcendental equation

$$\bar{z}_0 = \mathcal{E}_{z,0}\, \frac{Q'(f_0 - \bar{z}_0)}{Q(f_0 - \bar{z}_0)}. \tag{27 b}$$

Note that $\mathcal{E}_{z,0}$ is the variance of the normalized interference, i.e.

$$\mathcal{E}_{z,0} = \sum_{k=1}^{\infty} \mathcal{E}_{d,k} f_k^2 = \frac{1}{\mathcal{E}_\zeta} \sum_{\substack{i=-\infty \\ i\neq 0}}^{\infty} \mathcal{E}_{\alpha,i} h_i^2. \tag{28}$$

The last expression shows that $\mathcal{E}_{z,0}$ is equal to the *interference to noise ratio* (INR), i.e. the ratio of interference energy to Gaussian noise energy.

The numerical algorithms which solve Equation (27 b) require an initial value for the iteration. It can be shown, [16], that the root $\bar{z}_0$ is lower bounded by the following expression:

$$\bar{z}_0 \gtrapprox \frac{\mathcal{E}_{z,0}}{\mathcal{E}_{z,0} + 1}\, f_0. \tag{29}$$

This approximation turns out to be a good choice for the iteration initial value $z_{0,0}$ provided that the signal to interference and noise ratio (SINR) is high.

4. Second-Order Approximation

The product form of $Q(x)$ in Equation (12) is now truncated after the second term. This yields the approximation

$$Q(x) \approx Q(x_0)\, e^{c_1(x-x_0)}\, e^{c_2(x-x_0)^2} \tag{30}$$

where the parameters c_1 and c_2 are determined by Equation (13). The major problem with this expression is that a simple closed-form solution of the exponential interference moment $E[e^{z+z^2}]$ has not been found yet. This difficulty may be resolved by approximating the exponential e^{-x^2} with a more convenient expression. For example, the general shape of e^{-x^2} is similar to the cosine function around $x = 0$. In particular, the exponential may be upper bounded by

$$e^{-x^2} \leq \frac{2}{3} + \frac{1}{6}\exp(j\sqrt{6}x) + \frac{1}{6}\exp(-j\sqrt{6}x). \tag{31}$$

The parameters of the cosine function have been chosen such that the best approximation around $x = 0$ is obtained. Expanding both e^{-x^2} and the cosine function into a Taylor series, it appears that the first three non-zero terms are identical. The series become different only for sixth and higher orders in x.

The second-order approximation (30) and bound (31) are now used to determine the threshold probability P_{ex}. Using Equation (13) and the substitution $x_0 = f_0 - z_0$, the coefficients c_1 and c_2 are given by:

$$c_1 = -\lambda \tag{32}$$

$$c_2 = \frac{1}{2}\,\lambda(f_0 - z_0 - \lambda). \tag{33}$$

It can be shown, [16], that $c_2 < 0$. This guarantees that the second-order exponential is of the form e^{-x^2} with a negative exponent. Substituting Equations (30) and (31) as well as the relationships $x = f_0 - z$, $x_0 = f_0 - z_0$, $\lambda = -c_1$ and $\mu = \sqrt{-6c_2}$ into (11) results in

$$P_{\text{ex}} \approx Q(f_0 - z_0)\left\{\frac{2}{3}\, e^{-\lambda z_0}\, E\left[e^{\lambda z}\right] + \frac{1}{6}\, e^{-\Lambda z_0}\, E\left[e^{\Lambda z}\right] + \frac{1}{6}\, e^{-\Lambda^* z_0}\, E\left[e^{\Lambda^* z}\right]\right\} \tag{34}$$

where '*' denotes complex conjugation and $\Lambda \triangleq \lambda + j\mu$. The variable λ is defined by Equation (20), while μ is given by

$$\mu \triangleq \sqrt{-6c_2} = \sqrt{3\lambda(\lambda - f_0 + z_0)}. \tag{35}$$

The exponential moments of the interference may be expressed as a product of individual exponential exponents (14). Analogous to Section 3, the exact expression for the individual exponential moment (15) is used for the interference samples $f_k, \forall k \in \mathcal{I}_M$, with large magnitudes, while the small interference samples ($f_k, \forall k > M$) are bounded by Expression (17). This results in the second-order approximation of the threshold probability:

$$P_{\text{ex}} \approx \frac{2}{3}\, Q(f_0 - \bar{z}_0)\, \exp\left(\frac{1}{2}\,\mathcal{E}_{z,M}\bar{\lambda}^2 - \bar{\lambda}\bar{z}_0\right) \times \left\{ \prod_{k=1}^{M} \frac{\sinh\left(L_k \bar{\lambda} f_k\right)}{L_k\, \sinh\left(\bar{\lambda} f_k\right)} + \frac{1}{4}\, e^{-j\bar{\mu}\bar{z}_0} \prod_{k=1}^{M} \frac{\sinh\left(L_k \bar{\Lambda} f_k\right)}{L_k\, \sinh\left(\bar{\Lambda} f_k\right)} + \frac{1}{4}\, e^{j\bar{\mu}\bar{z}_0} \prod_{k=1}^{M} \frac{\sinh\left(L_k \bar{\Lambda}^* f_k\right)}{L_k\, \sinh\left(\bar{\Lambda}^* f_k\right)} \right\}. \tag{36}$$

In order to obtain a good approximation, the value $z_0 = \bar{z}_0$ has to be chosen carefully. It is known from the previous section that the best first-order bound is obtained when the parameter $\bar{z}_0$ satisfies condition (26 b). Since the second-order product representation is a better approximation of $Q(x)$ than the first-order expansion (18), the expression (36) provides a better estimation of P_{ex} than the respective first-order formula (26). Therefore, the same optimization criterion is used as in the first-order case, and the parameter $\bar{z}_0$ is chosen such that it satisfies conditions (26 b) and (25). $\bar{\lambda}$ is obtained by substituting $\bar{z}_0$ for z_0 into Equation (20). The parameter $\bar{\Lambda}$ is then obtained through

$$\bar{\Lambda} = \bar{\lambda} + j\bar{\mu} \tag{37}$$

$$\bar{\mu} = \sqrt{3\bar{\lambda}(\bar{\lambda} - f_0 + \bar{z}_0)}. \tag{38}$$

4.1. Special Case: Binary Modulation

In the case of binary modulation there are only two modulation levels, i.e. $L_k = 2$ and $d_k \in \{-1, 1\}$, $\forall k \in \mathbb{N}_0$. The derived bounds on the error probability can be simplified by using the hyperbolic trigonometric relationships

$$\frac{1}{2}\,\frac{\sinh(2x)}{\sinh(x)} = \cosh(x) \tag{39}$$

$$2\,\coth(2x) - \coth(x) = \tanh(x). \tag{40}$$

5. Numerical Results and Comparison

For the presentation of our results, we consider a single user communications system using binary modulation, i.e. $L_i = 2, \mathcal{A}_i = \{-1; 1\}$, $\forall i \in \mathbb{Z}$. The impulse response of the overall channel is modeled by the well known Chebyshev pulse [3]

$$h(t) = \sum_{i=1}^{2} A_i \cos\left(\omega_i |t|/T - \phi_i\right)\, \exp\left(-\beta_i |t|/T\right) \tag{41}$$

$$\begin{array}{llll} A_1 = 0.4032 & \omega_1 = 2.839 & \phi_1 = 0.7553 & \beta_1 = 0.4587 \\ A_2 = 0.7163 & \omega_2 = 1.176 & \phi_2 = 0.1602 & \beta_2 = 1.107. \end{array}$$

Sampling the received signal at a period T yields the channel samples $h_i = h(t_0 + iT)$, where t_0/T is the *relative sampling instant*. For practical purposes, we restrict the number of nonzero interference samples to 1000, which means that $h_i = 0$, $\forall |i| > 500$. Note that the signal to noise ratio (SNR) of this model is SNR $= h^2(0)/\mathcal{E}_\zeta$.

Instead of the exact exhaustive method, the very accurate and much more efficient algorithm of Helstrom [7] is used as a reference algorithm for comparison with our approximations. For the following results, the relative error of P_{ex} obtained with Helstrom's method was forced to be below 10^{-10}. Obviously, the feature of preselecting an arbitrary accuracy is highly desirable. It comes, however, at the price of significantly increased computational effort when compared to the approximations described in Sections 3 and 4.

The performance of our approximations is evaluated using the *relative error* between the value $\tilde{P}_{\text{ex}}$ obtained from the approximation and the "exact" threshold probability P_{ex} provided by Helstrom's algorithm:

$$\varepsilon \triangleq \frac{\tilde{P}_{\text{ex}} - P_{\text{ex}}}{\min\left\{\tilde{P}_{\text{ex}}; P_{\text{ex}}\right\}}. \tag{42}$$

The "minimum" normalization in this definition ensures that too large and too small approximations are weighted equally.

Contour plots will be used for the presentation of our results. The regions of constant magnitudes $|\varepsilon|$, represented by contour lines, are plotted over the surface spanned by the SNR (abscissa) and the relative sampling instant t_0/T (ordinate). The ratio between two adjacent contour lines is always a factor of ten. The values printed on the lines represent the logarithm of the respective value to the base 10, i.e. a contour line with the value '-2' shows the region of constant values $P_{\text{ex}} = 10^{-2}$ or $|\varepsilon| = 10^{-2}$, respectively.

Figure 1 shows four contour plots. The first plot (a) represents the values of the "exact" threshold probability P_{ex} as calculated with Helstrom's algorithm. The other three plots display the magnitude of the relative errors ($\log_{10}|\varepsilon|$) for (b) the energy bound (27), (c) the first-order bound (26) and (d) the second-order approximation (36).

Figure 1(a) shows that the threshold probability starts to decrease sharply when the SNR exceeds 10 dB provided that the relative sampling instant t_0/T is below 0.2. For larger t_0/T, low error rates may not be achieved, even if the SNR becomes very large, because the interference dominates the behavior.

The energy upper bound in Figure 1(b) performs unsatisfactorily when the exact threshold probability P_{ex} is very low, i.e. for high SNR's greater than 20 dB and $t_0/T < 0.2$. In this region, the estimate of the bound is too large by several orders of magnitude, and the results are overly pessimistic. The same qualitative behavior is found for other

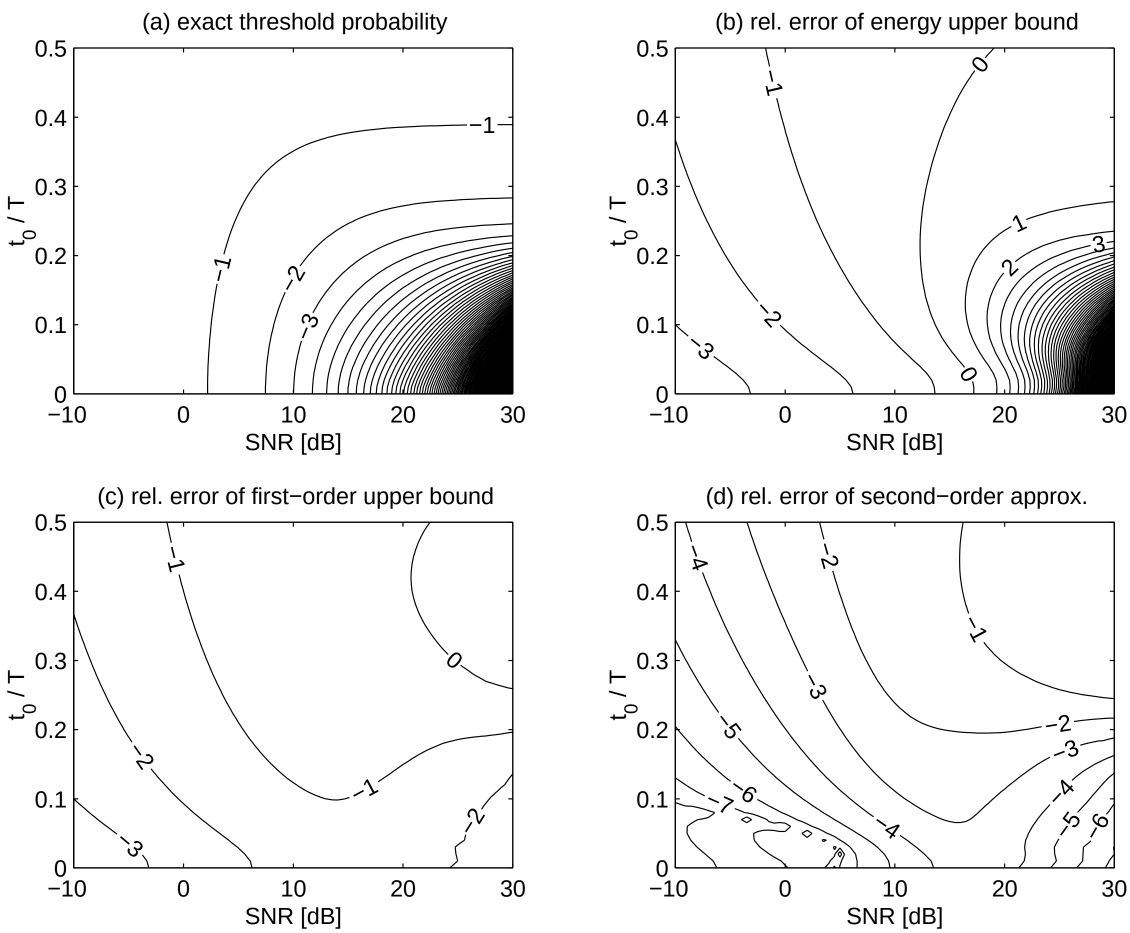

Figure 1. Contour plots for the Chebyshev pulse showing the exact threshold probability $\log_{10} P_{\text{ex}}$ (a), and the relative errors $\log_{10} |\varepsilon|$ for several bounds and approximations (b) – (d).

channel waveforms as well and is, in fact, a characteristic of all "energy" approximations such as the Saltzberg bound [2, 13] or the Gaussian approximation, which take into account only the variance of the interference [16].

The more accurate algorithms (Figures 1(c) and (d)) perform good to excellent for all values of the SNR and sampling instant considered here. In general, the second-order approximation yields more accurate results than the first-order bound. Even in the most extreme situations of high SNR's and large relative sampling instants, the relative error never exceeded $100\ \%$ for the second-order approximation.

6. Conclusion

We have introduced new strict and approximate upper bounds on the error probability which are computationally efficient. Our approximations include an arbitrary parameter that is optimized in order to achieve the tightest bound. The resulting exponential moments of the interference can be computed easily [15]. For the calculation of the approximations, it is simply required to find numerically the root of a transcendental equation and to evaluate exponential moments. An additional feature is that small interference components can be tightly upper bounded by an expression involving only their combined variance.

Terminating the Taylor series of $\ln Q(x)$ after the linear term results in the first-order upper bound. A special case of it, the energy upper bound, requires knowledge about only the variance (energy) of the interference and the variance of the Gaussian noise. This bound provides reasonable accuracy for low to moderate SNR's. However, its results are extremely pessimistic when the true error probability is very small ($< 10^{-3}$).

The general first-order upper bound requires explicit knowledge about the strong interference samples. As a consequence, it provides significantly better results especially in situations where the energy bound fails. For the Chebyshev pulse and an open eye, the relative errors between the first-order bound and the exact error probability did not exceed $20\ \%$. This number increased to a maximum of $171\ \%$ when the eye was closed.

The second-order approximation considers in addition the quadratic term of the Taylor series and replaces the resulting factor e^{-x^2} with a cosine function. It is the most accurate albeit most complex of the derived algorithms. For the Chebyshev pulse, the relative errors were always below $1\ \%$ (open eye) and $82\ \%$ (closed eye). Similar results have been obtained for the Gaussian pulse and the ideal bandwidth limited (sinc) pulse [16]. The price to be paid is an increased amount of necessary operations. Instead of none (energy bound) or one real exponential moment (first-order bound), it requires the evaluation of one real and one complex exponential moment. This is comparable to the complexity of Yue's approximation [11]. In addition, the proposed and Yue's method yield comparable accuracies. The difference is that Yue's results turn out to be below the true error probability in most cases while we derived strict and approximate upper bounds.

The presented bounds are significantly more efficient than the more accurate approximations of Helstrom [7] and Beaulieu [10]. Helstrom's algorithm, for example, performs the calculation of an inverse Laplace transformation by numerical quadrature and requires the evaluation of one complex exponential moment at each integration point. Depending on the system parameters and the desired accuracy, the number of necessary integration points can vary between ten and several hundred. Beaulieu's approximation has a similar degree of computational complexity.

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